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Generating sequences and semigroups of valuations on 2-dimensional normal local rings

Arpan Dutta [(1)]

Abstract. — In this paper we develop a method for computing valuation semigroups for valuations dominating the ring of a two dimensional quotient singularity. Suppose that K is an algebraically closed field of characteristic zero, $K[X,Y]$ is a polynomial ring over K and ν is a rational rank 1 non discrete valuation of the field $K(X,Y)$ which dominates $K[X,Y]_{(X,Y)}$. Given a finite abelian group H acting diagonally on $K[X,Y]$, and a generating sequence of ν in $K[X,Y]$ whose members are eigenfunctions for the action of H, we compute the semigroup $S^{K[X,Y]^H}(\nu)$ of values of elements of $K[X,Y]^H$. We further determine when $S^{K[X,Y]}(\nu)$ is a finitely generated $S^{K[X,Y]^H}(\nu)$-module.

Résumé. — Dans cet article, nous développons une méthode de calcul de semigroupes d'évaluation pour les évaluations dominant l'anneau d'une singularité de quotient à deux dimensions. Supposons que K est un corps algébriquement clos de caractéristique zéro, $K[X,Y]$ est un anneau polynomial sur K et ν est une évaluation rationnelle non discrète de rang 1 du corps $K(X,Y)$ qui domine $K[X,Y]_{(X,Y)}$. Étant donné un groupe H abelien fini agissant en diagonale sur $K[X,Y]$ et une suite génératrice de ν dans $K[X,Y]$ dont les membres sont des fonctions propres pour l'action de H, nous calculons le semigroupe $S^{K[X,Y]^H}(\nu)$ de valeurs d'éléments de l'anneau invariant $K[X,Y]^H$. Nous déterminons en outre quand $S^{K[X,Y]}(\nu)$ est un $S^{K[X,Y]^H}(\nu)$-module de type fini.

Notations

Let $\mathbb{N}$ denotes the natural numbers $\{0,1,2,\cdots\}$. We denote the positive integers by $\mathbb{Z}_{>0}$ and the positive rational numbers by $\mathbb{Q}_{>0}$. If the greatest common divisor of two positive integers a and b is d, this is denoted by $(a,b)=d$. If $\{\gamma_k\}_{k\geqslant 0}$ is a set of rational numbers, we define $G(\gamma_0,\cdots,\gamma_n)=\sum_{k=0}^{n}\gamma_k\mathbb{Z}$ and $G(\gamma_0,\gamma_1,\cdots)=\sum_{k\geqslant 0}\gamma_k\mathbb{Z}$. Similarly we define $S(\gamma_0,\cdots,\gamma_n)=\sum_{k=0}^{n}\gamma_k\mathbb{N}$ and $S(\gamma_0,\gamma_1,\cdots)=\sum_{k\geqslant 0}\gamma_k\mathbb{N}$. If a group G is generated by $g_1,\cdots,g_n$, we denote this by $G=<g_1,\cdots,g_n>$.

(1) Department of Mathematics, IISER Mohali, Knowledge City, Sector 81, Manauli PO, SAS Nagar, Punjab, 140306 (India) — arpandutta@iisermohali.ac.in

Introduction

Let R be a local domain with maximal ideal m_R and quotient field L, and ν be a valuation of K which dominates R. Let V_ν be the valuation ring of ν, with maximal ideal m_ν and Φ_ν be the valuation group of ν. The associated graded ring of R along the valuation ν, defined by Teissier in [[14]] and [[15]], is

$$\mathrm{gr}_\nu(R) = \bigoplus_{\gamma\in\Phi_\nu} \mathcal{P}_\gamma(R)/\mathcal{P}^+_\gamma(R) \tag{0.1}$$

where

$$\mathcal{P}_\gamma(R) = \{f \in R \mid \nu(f) \geqslant \gamma\} \text{ and } \mathcal{P}^+_\gamma(R) = \{f \in R \mid \nu(f) > \gamma\}.$$

In general, $\mathrm{gr}_\nu(R)$ is not Noetherian. The valuation semigroup of ν on R is

$$S^R(\nu) = \{\nu(f) \mid f \in R \setminus (0)\}. \tag{0.2}$$

If $R/m_R = V_\nu/m_\nu$ then $\mathrm{gr}_\nu(R)$ is the group algebra of $S^R(\nu)$ over R/m_R, so that $\mathrm{gr}_\nu(R)$ is completely determined by $S^R(\nu)$.

A generating sequence of ν in R is a set of elements of R whose classes in $\mathrm{gr}_\nu(R)$ generate $\mathrm{gr}_\nu(R)$ as an R/m_R-algebra. An important problem is to construct a generating sequence of ν in R which gives explicit formulas for the value of an arbitrary element of R, and gives explicit computations of the algebra (0.1) and the semigroup (0.2). For regular local rings R of dimension 2, the construction of generating sequences is realized in a very satisfactory way by Spivakovsky [[13]] (with the assumption that R/m_R is algebraically closed) and by Cutkosky and Vinh [[6]] for arbitrary regular local rings of dimension 2. A consequence of this theory is a simple classification of the semigroups which occur as a valuation semigroup on a regular local ring of dimension 2. There has been some success in constructing generating sequences in Noetherian local rings of dimension $\geqslant 3$, for instance in [[7]], [[10]], [[11]] and [[15]], but the general situation is very complicated and is not well understood.

Another direction is to construct generating sequences in normal 2 dimensional Noetherian local rings. This is also extremely difficult. In Section 9 of [[6]], a generating sequence is constructed for a rational rank 1 non discrete valuation in the ring $R = k[u, v, w]/(uv - w^2)$, from which the semigroup is constructed. The example shows that the valuation semigroups of valuations dominating a normal two dimensional Noetherian local ring are much more complicated than those of valuations dominating a two dimensional regular local ring. In this thesis, we develop the method of this example into a general theory.

If R is a 2 dimensional Noetherian local domain, and ν is a valuation of the quotient field L of R which dominates R, it follows from Abhyankar's inequality [[1]] that the valuation group Φ_ν of ν is a finitely generated group, except in the case when the rational rank of ν is 1 ($\Phi_\nu \otimes \mathbb{Q} \cong \mathbb{Q}$) and Φ_ν is non discrete. As this is the essentially difficult case in dimension 2, we will restrict to such valuations.

Let K be an algebraically closed field of characteristic 0 and $K[X, Y]$ be a polynomial ring in two variables, which has the maximal ideal $\mathfrak{m} = (X, Y)$. Let $\alpha \in K$ be a primitive m-th root of unity and $\beta \in K$ be a primitive n-th root of unity. Now the group $\mathbb{U}_m \times \mathbb{U}_n$ acts on $K[X, Y]$ by K-algebra isomorphisms, where

$$(\alpha^i, \beta^j)X = \alpha^i X \text{ and } (\alpha^i, \beta^j)Y = \beta^j Y.$$

In Theorem 1.2, we give a classification of the subgroups $H_{i,j,t,x}$ of $\mathbb{U}_m \times \mathbb{U}_n$. In Remark 1.3 we observe that without any loss of generality, we can assume $i = j = 1$ and $H = H_{1,1,t,x}$ is a subdirect product of $\mathbb{U}_m \times \mathbb{U}_n$. Let

$$A = K[X, Y]^H \text{ and } \mathfrak{n} = \mathfrak{m} \cap A.$$

We say that $f \in K[X, Y]$ is an eigenfunction for the action of H on $K[X, Y]$ if for all $g \in H$, $gf = \lambda_g f$ for some $\lambda_g \in K$. Throughout the paper, we use the expression $\forall \ b \equiv ax (\mathrm{mod}\ t)$ as an abbreviation for the following expression,

$$\forall \ a, b \in \mathbb{Z} \text{ such that } b \equiv ax (\mathrm{mod}\ t).$$

Let ν be a rational rank 1 non discrete valuation dominating the regular local ring $K[X, Y]_{\mathfrak{m}}$. Using the algorithm of [13] or [6], we construct a generating sequence

$$Q_0 = X, Q_1 = Y, Q_2, \ldots \tag{0.3}$$

of ν in $K[X, Y]$. Let ν^* be the restriction of ν to the quotient field of A. In Theorem 3.1, we give an explicit computation of the valuation semigroups $S^{A_{\mathfrak{n}}}(\nu)$, when the members of the generating sequence (0.3) are eigenfunctions for the action of H on $K[X, Y]$.

Suppose that a Noetherian local domain B dominates a Noetherian local domain A. Let L be the quotient field of A, M be the quotient field of B and suppose that M is finite over L. Suppose that ω is a valuation of L which dominates A and ω^* is an extension of ω to M which dominates B. We can ask if $\mathrm{gr}_{\omega^*}(B)$ is a finitely generated $\mathrm{gr}_\omega(A)$-module or if $S^B(\omega^*)$ is a finitely generated $S^A(\omega)$-module. In general, $\mathrm{gr}_{\omega^*}(B)$ is not a finitely generated $\mathrm{gr}_\omega(A)$-algebra, so is certainly not a finitely generated $\mathrm{gr}_\omega(A)$-module. However, is is shown in Theorem 1.5. [[5]] that if A and B are essentially of finite type over a field characteristic zero, then there exists a birational extension A_1 of A and a birational extension B_1 of B such that ω^* dominates B_1, ω dominates A_1, B_1 dominates A_1 and $\text{gr}_{\omega^*}(B_1)$ is a finitely generated $\text{gr}_\omega(A_1)$-module (so $S^{B_1}(\omega^*)$ is a finitely generated $S^{A_1}(\omega)$-module).

The situation is much more subtle in positive characteristic and mixed characteristic. In Theorem 1 [[4]], it is shown that If A and B are excellent of dimension two and $L \to M$ is separable, then there exist birational extension A_1 of A and B_1 of B such that A_1 and B_1 are regular, B_1 dominates A_1, ω^* dominates B_1 and $\text{gr}_{\omega^*}(B_1)$ is a finitely generated $\text{gr}_\omega(A_1)$-algebra if and only if the valued field extension $L \to M$ is without defect. For a discussion of defect in a finite extension of valued fields, see [[8]].

In this paper, we completely answer the question of finite generation of $S^{K[X,Y]_\mathfrak{m}}(\nu)$ as a $S^{A_\mathfrak{n}}(\nu)$-module (and hence of $\text{gr}_\nu(K[X,Y]_\mathfrak{m})$ as a $\text{gr}_\nu(A_\mathfrak{n})$-module) for valuations with a generating sequence of eigenfunctions. We obtain the following results in Section 4.

Proposition 0.1. — *Let $R_\mathfrak{m} = K[X,Y]_{(X,Y)}$ and H be a subdirect product of $\mathbb{U}_m \times \mathbb{U}_n$. Let ν be a rational rank 1 non discrete valuation ν dominating $R_\mathfrak{m}$ with a generating sequence (0.3) of eigenfunctions for H. Then $S^{R_\mathfrak{m}}(\nu)$ is finitely generated over the subsemigroup $S^{A_\mathfrak{n}}(\nu)$ if and only if $\exists\ N \in \mathbb{Z}_{>0}$ such that $Q_r \in A\ \forall\ r \geqslant N$. Further, if $Q_N \in A$, then $Q_M \in A\ \forall\ M \geqslant N \geqslant 1$.*

Theorem 0.2. — *Let $R_\mathfrak{m} = K[X,Y]_{(X,Y)}$ and H be a subdirect product of $\mathbb{U}_m \times \mathbb{U}_n$.*

1) *$\exists$ a rational rank 1 non discrete valuation ν dominating $R_\mathfrak{m}$ with a generating sequence (0.3) of eigenfunctions for $H \iff (m,n) = t$.*
2) *If $(m,n) = t = 1$, then $S^{R_\mathfrak{m}}(\nu)$ is a finitely generated $S^{A_\mathfrak{n}}(\nu)$-module for all rational rank 1 non discrete valuations ν which dominate $R_\mathfrak{m}$ and have a generating sequence (0.3) of eigenfunctions for H.*
3) *If $(m,n) = t > 1$, then $S^{R_\mathfrak{m}}(\nu)$ is not a finitely generated $S^{A_\mathfrak{n}}(\nu)$-module for all rational rank 1 non discrete valuations ν which dominate $R_\mathfrak{m}$ and have a generating sequence (0.3) of eigenfunctions for H.*

In Section 5, we show that for the valuations we consider, the restriction of ν to the quotient field of A does not split in $K[X,Y]_\mathfrak{m}$. The failure of non splitting can be an obstruction to finite generation of $S^B(\omega^*)$ as an $S^A(\omega)$-module (Theorem 5 [[4]]), but our result shows that it is not a sufficient condition.

1. Subgroups of $U_m \times U_n$

Let K be an algebraically closed field of characteristic zero. Let α be a primitive m-th root of unity, and β be a primitive n-th root of unity, in K. We denote $\mathbb{U}_m =< \alpha >$, and $\mathbb{U}_n =< \beta >$, which are multiplicative cyclic groups of orders m and n respectively.

LEMMA 1.1 (Goursat). — *Let A and B be two groups. There is a bijective correspondence between subgroups $G \leqslant A \times B$, and 5-tuples $\{\overline{G_1}, G_1, \overline{G_2}, G_2, \theta\}$, where*

$$G_1 \trianglelefteq \overline{G_1} \leqslant A\,,\; G_2 \trianglelefteq \overline{G_2} \leqslant B\,,\; \theta : \frac{\overline{G_1}}{G_1} \to \frac{\overline{G_2}}{G_2} \text{ is an isomorphism.}$$

THEOREM 1.2. — *Given positive integers i, j, t, x satisfying the given conditions*

$$i|m,\; j|n,\; t|\frac{m}{i},\; t|\frac{n}{j},\; (x,t) = 1,\; 1 \leqslant x \leqslant t$$

let

$$H_{i,j,t,x} = \{(\alpha^{ai}, \beta^{bj}) \,|\, b \equiv ax(mod\ t)\}. \tag{1.1}$$

Then the $H_{i,j,t,x}$ are subgroups of $\mathbb{U}_m \times \mathbb{U}_n$. And given any subgroup G of $\mathbb{U}_m \times \mathbb{U}_n$, there exist unique i, j, t, x satisfying the above conditions such that $G = H_{i,j,t,x}$.

Proof. — We first show that the condition $b \equiv ax(\text{mod } t)$ is well defined under the given conditions on i, j, t, x. Suppose $(\alpha^{a_1 i}, \beta^{b_1 j}) = (\alpha^{a_2 i}, \beta^{b_2 j})$, that is, $a_1 i \equiv a_2 i(\text{mod } m)$, and $b_1 j \equiv b_2 j(\text{mod } n)$. Then, $\frac{m}{i} \mid (a_1 - a_2)$ and $\frac{n}{j} \mid (b_1 - b_2)$. Thus, $t \mid (a_1 - a_2)$ and $t \mid (b_1 - b_2)$, hence $t \mid (b_1 - b_2) - (a_1 - a_2)x$. So, $[b_1 - a_1 x] \equiv [b_2 - a_2 x](\text{mod } t)$.

We now show $H_{i,j,t,x}$ is a subgroup of $\mathbb{U}_m \times \mathbb{U}_n$. Taking $a = b = 0$, we have $(1, 1) \in H_{i,j,t,x}$. Let $(\alpha^{ai}, \beta^{bj})$, $(\alpha^{ci}, \beta^{dj}) \in H_{i,j,t,x}$ be distinct elements. Then $b \equiv ax(\text{mod t})$, and $d \equiv cx(\text{mod t})$. Hence $(b - d) \equiv (a - c)x(\text{ mod t})$. So, $(\alpha^{(a-c)i}, \beta^{(b-d)j}) = (\alpha^{ai}, \beta^{bj})(\alpha^{ci}, \beta^{dj})^{-1} \in H_{i,j,t,x}$. Hence $H_{i,j,t,x}$ is a subgroup.

By Goursat's Lemma, the subgroups of $\mathbb{U}_m \times \mathbb{U}_n$ are in bijective correspondence with the 5-tuples $\{\overline{G_1}, G_1, \overline{G_2}, G_2, \theta\}$, where $G_1 \trianglelefteq \overline{G_1} \leqslant \mathbb{U}_m$, $G_2 \trianglelefteq \overline{G_2} \leqslant \mathbb{U}_n$, $\theta : \frac{\overline{G_1}}{G_1} \simeq \frac{\overline{G_2}}{G_2}$. Now any subgroup of $\mathbb{U}_m =< \alpha >$ is of the form $H_i =< \alpha^i >= \mathbb{U}_{\frac{m}{i}}$, where $i|m$. Since H_i is an abelian group, any subgroup is normal. Any subgroup of H_i is of the form $H_{it_i} =< \alpha^{it_i} >= \mathbb{U}_{\frac{m}{it_i}}$, where $t_i|\frac{m}{i}$. Similarly, any subgroup of $\mathbb{U}_n$ is of the form $H_j =< \beta^j >= \mathbb{U}_{\frac{n}{j}}$, where $j|n$. And any subgroup of H_j is of the form $H_{jt_j} =< \beta^{jt_j} >= \mathbb{U}_{\frac{n}{jt_j}}$, where $t_j|\frac{n}{j}$. Now, $\frac{\mathbb{U}_{\frac{m}{i}}}{\mathbb{U}_{\frac{m}{it_i}}} \simeq \mathbb{U}_{t_i}$ and $\frac{\mathbb{U}_{\frac{n}{j}}}{\mathbb{U}_{\frac{n}{jt_j}}} \simeq \mathbb{U}_{t_j}$. So, $\theta_{ij} : \frac{\mathbb{U}_{\frac{m}{i}}}{\mathbb{U}_{\frac{m}{it_i}}} \simeq \frac{\mathbb{U}_{\frac{n}{j}}}{\mathbb{U}_{\frac{n}{jt_j}}} \iff t_i = t_j$.

Define $t = t_i = t_j$. Thus the subgroups of $\mathbb{U}_m \times \mathbb{U}_n$ are in bijective correspondence with the set of 5-tuples,

$$\begin{aligned}&(<\alpha^{it}>, <\alpha^{i}>, <\beta^{jt}>, <\beta^{j}>, \theta_{ij})\\ &\text{where } i|m,\ j|n,\ t|\frac{m}{i},\ t|\frac{n}{j} \text{ and } \theta_{ij} : \frac{<\alpha^{i}>}{<\alpha^{it}>} \simeq \frac{<\beta^{j}>}{<\beta^{jt}>}.\end{aligned} \tag{1.2}$$

Any such isomorphism is given by $\theta_{ij}(\overline{\alpha^i}) = \overline{\beta^{xj}}$, where $(x,t) = 1$, $1 \leqslant x \leqslant t$, and $\overline{v}$ denotes the residue of an element $v \in <\alpha^i>$ in $\frac{<\alpha^i>}{<\alpha^{it}>}$, or the residue of an element $v \in <\beta^j>$ in $\frac{<\beta^j>}{<\beta^{jt}>}$.
If $G_{\theta_{ij}}$ denotes the graph of θ_{ij}, then $G_{\theta_{ij}} = \{(\overline{\alpha^{ri}}, \overline{\beta^{rxj}}) | r \in \mathbb{N}\}$. Denoting the natural surjection $p : <\alpha^i> \times <\beta^j> \longrightarrow \frac{<\alpha^i>}{<\alpha^{it}>} \times \frac{<\beta^j>}{<\beta^{jt}>}$, we have

$$\begin{aligned}p^{-1}(G_{\theta_{ij}}) &= \{(\alpha^{ai}, \beta^{bj}) \,|\, \alpha^{\overline{ai}} = \alpha^{\overline{ri}}, \beta^{\overline{bj}} = \beta^{\overline{rxj}},\ \text{for some } r \in \mathbb{N}\}\\ &= \{(\alpha^{ai}, \beta^{bj}) \,|\, \alpha^{(a-r)i} \in <\alpha^{it}>, \beta^{(b-rx)j} \in <\beta^{jt}>,\ \text{for some } r \in \mathbb{N}\}\\ &= \{(\alpha^{ai}, \beta^{bj}) \,|\, a \equiv r(\text{mod t}), b \equiv rx(\text{mod t}),\ \text{for some } r \in \mathbb{N}\}.\end{aligned}$$

We now show that,

$$a \equiv r(\text{mod t}), b \equiv rx(\text{mod t}),\ \text{for some } r \in \mathbb{N} \Longleftrightarrow b \equiv ax(\text{mod t}). \tag{1.3}$$

If $a \equiv r(\text{mod t}), b \equiv rx(\text{mod t})$, then $a - r = td$ for some integer d. Then $b - ax = b - (td + r)x \equiv b - rx(\text{mod t}) \equiv 0(\text{mod t}) \Longrightarrow b \equiv ax(\text{mod t})$. Conversely if $b \equiv ax(\text{mod t})$, and $a \equiv r(\text{mod t})$ for some r, then $b \equiv rx(\text{mod t})$. Thus we have established (1.3). So, $p^{-1}(G_{\theta_{ij}}) = \{(\alpha^{ai}, \beta^{bj}) \,|\, b \equiv ax(\text{mod t})\}$. Thus we have that any subgroup of $\mathbb{U}_m \times \mathbb{U}_n$ is of the form

$$H_{i,j,t,x} = \{(\alpha^{ai}, \beta^{bj}) \,|\, b \equiv ax(\text{mod t})\,;\, i|m,\ j|n,\ t|\frac{m}{i},\ t|\frac{n}{j},\ (x,t) = 1,\\ 1 \leqslant x \leqslant t\}.$$

We now establish uniqueness. Let (i_1, j_1, t_1, x_1) and (i_2, j_2, t_2, x_2) be two distinct quadruples satisfying the conditions of the theorem, such that $H_{i_1,j_1,t_1,x_1} = H_{i_2,j_2,t_2,x_2}$. From (1.2), we observe $H_{i_1,j_1,t_1,x_1} = H_{i_2,j_2,t_2,x_2}$ implies

$$\begin{aligned}&(<\alpha^{i_1 t_1}>, <\alpha^{i_1}>, <\beta^{j_1 t_1}>, <\beta^{j_1}>, \theta^{(1)}_{i_1 j_1})\\ &\qquad = (<\alpha^{i_2 t_2}>, <\alpha^{i_2}>, <\beta^{j_2 t_2}>, <\beta^{j_2}>, \theta^{(2)}_{i_2 j_2}).\end{aligned}$$

Now, $<\alpha^{i_1}> = <\alpha^{i_2}> \Longrightarrow |<\alpha^{i_1}>| = |<\alpha^{i_2}>| \Longrightarrow m/i_1 = m/i_2 \Longrightarrow i_1 = i_2 = i$. And, $<\alpha^{it_1}> = <\alpha^{it_2}> = m/it_1 = m/it_2 = t_1 = t_2 = t$.Similarly $j = j_1 = j_2$. Now, $\theta^{(1)}_{ij} = \theta^{(2)}_{ij} \Longrightarrow \theta^{(1)}_{ij}(\overline{\alpha^i}) = \theta^{(2)}_{ij}(\overline{\alpha^i}) \Longrightarrow \overline{\beta^{x_1 j}} = \overline{\beta^{x_2 j}}$ in $\frac{<\beta^j>}{<\beta^{tj}>}$. Thus, $t \mid |x_1 - x_2|$. Since $0 < x_1, x_2 \leqslant t$, we have $|x_1 - x_2| = 0$, i.e. $x_1 = x_2$. Let $x = x_1 = x_2$. Then $(i, j, t, x) = (i_1, j_1, t_1, x_1) = (i_2, j_2, t_2, x_2)$ is unique. □

We observe $H_{i,j,t,x} = \{((\alpha^i)^a, (\beta^j)^b) \mid b \equiv ax (\text{mod } t)\} \leqslant \mathbb{U}_{\frac{m}{i}} \times \mathbb{U}_{\frac{n}{j}}$. Since $(x, t) = 1$, $H_{i,j,t,x}$ is a subdirect product of $\mathbb{U}_{\frac{m}{i}} \times \mathbb{U}_{\frac{n}{j}}$. So without loss of generality we can assume $i = j = 1$, that is, $H_{1,1,t,x}$ is a subdirect product of $\mathbb{U}_m \times \mathbb{U}_n$. For the rest of the paper, we adopt the following notation,

Remark 1.3. — $H = H_{1,1,t,x}$ is a subdirect product of $\mathbb{U}_m \times \mathbb{U}_n$. Thus $H = \{(\alpha^a, \beta^b) \mid b \equiv ax (\text{mod } t)\}$, where $t \mid m$, $t \mid n$, $(x, t) = 1$ and $1 \leqslant x \leqslant t$.

Proposition 1.4. — *Let H be as in Remark 1.3. Write $m = Mt$ and $n = Nt$ where $M, N \in \mathbb{Z}_{>0}$. Then $|H| = MNt$.*

Proof. — Recall, $H = \{(\alpha^a, \beta^b) \mid b \equiv ax (\text{mod } t)\}$. We observe, as elements of H, $(\alpha^{a_1}, \beta^{b_1}) = (\alpha^{a_2}, \beta^{b_2})$ if and only if $a_1 \equiv a_2 (\text{mod } Mt)$ and $b_1 \equiv b_2 (\text{mod } Nt)$. Thus every element of H has an unique representation,

$$H = \{(\alpha^a, \beta^b) \mid b \equiv ax (\text{mod } t),\ 0 \leqslant a < Mt,\ 0 \leqslant b < Nt\}. \tag{1.4}$$

Hence there is a bijective correspondence,

$$\begin{aligned} H &\longleftrightarrow \{(a, b) \mid b \equiv ax (\text{mod } t),\ 0 \leqslant a < Mt,\ 0 \leqslant b < Nt,\ a, b \in \mathbb{Z}\} \\ &\longleftrightarrow \{(a, ax + \lambda t) \mid 0 \leqslant a < Mt,\ 0 \leqslant ax + \lambda t < Nt,\ a, \lambda \in \mathbb{Z}\} \\ &\longleftrightarrow \{(a, \lambda) \mid 0 \leqslant a < Mt,\ 0 \leqslant \lambda + \frac{ax}{t} < N,\ a, \lambda \in \mathbb{Z}\}. \end{aligned}$$

Hence there are Mt possible choices for a. And for each choice of a, there are N possible choices for λ. Thus $|H| = MNt$. □

2. Generating Sequences

In this section we establish notation which will be used throughout the paper. Let $R = K[X, Y]$ be a polynomial ring in two variables over an algebraically closed field K of characteristic zero. Let $\mathfrak{m} = (X, Y)$ be the maximal ideal of R. Then $\mathbb{U}_m \times \mathbb{U}_n$ acts on R by K-algebra isomorphisms satisfying

$$(\alpha^x, \beta^y) \cdot (X^r Y^s) = \alpha^{rx} \beta^{sy} X^r Y^s. \tag{2.1}$$

Thus, $R^H = \{\sum_{r,s} c_{r,s} X^r Y^s \in R \,|\, \alpha^{ra} \beta^{sb} = 1\ \forall\ r, s,\ \forall\ b \equiv ax (\text{mod t})\}$.
$f \in R$ is defined to be an eigenfunction of H if $(\alpha^a, \beta^b) \cdot f = \lambda_{ab} f$ for some $\lambda_{ab} \in K$, for all $(\alpha^a, \beta^b) \in H$. Eigenfunctions of H are of the form $f = \sum_{r,s} c_{r,s} X^r Y^s \in R$ such that $\alpha^{ra} \beta^{sb}$ is a common constant $\forall\ r, s$ such that $c_{r,s} \neq 0, \forall\ b \equiv ax (\text{mod t})$.

Let ν be a rational rank 1 non discrete valuation of $K(X,Y)$ which dominates $R_{\mathfrak{m}}$. The algorithm of Theorem 4.2 of [[6]] (as refined in Section (8) of [[6]]) produces a generating sequence

$$Q_0 = X, Q_1 = Y, Q_2, \cdots \tag{2.2}$$

of elements in R which have the following properties.

1) Let $\gamma_l = \nu(Q_l)\ \forall\ l \geqslant 0$ and $\overline{m_l} = [G(\gamma_0, \cdots, \gamma_l) : G(\gamma_0, \cdots, \gamma_{l-1})] = \min\{q \in \mathbb{Z}_{>0} \mid q\gamma_l \in G(\gamma_0, \cdots, \gamma_{l-1})\}\ \forall\ l \geqslant 1$. Then $\gamma_{l+1} > \overline{m_l}\gamma_l\ \forall\ l \geqslant 1$.
2) Set $d(l) = \deg_Y(Q_l)\ \forall\ l \in \mathbb{Z}_{>0}$. Then, $Q_l = Y^{d(l)} + Q_l^*(X,Y)$, where $\deg_Y(Q_l^*(X,Y)) < d(l)$. We have that, $d(1) = 1$, $d(l) = \prod_{k=1}^{l-1} \overline{m_k}\ \forall\ l \geqslant 2$. In particular, $1 \leqslant l_1 \leqslant l_2 \implies d(l_1) \mid d(l_2)$.
3) Every $f \in R$ with $\deg_Y(f) = d$ has a unique expression

$$f = \sum_{m=0}^{d} [(\sum_l b_{l,m} X^l) Q_1^{j_1(m)} \cdots Q_r^{j_r(m)}]$$

where $b_{l,m} \in K, 0 \leqslant j_l(m) < \overline{m_l}\ \forall\ l \geqslant 1$, and $\deg_Y[Q_1^{j_1(m)} \cdots Q_r^{j_r(m)}] = m\ \forall\ m$. Writing $f_m = (\sum_l b_{l,m} X^l) Q_1^{j_1(m)} \cdots Q_r^{j_r(m)}$, we have that $\nu(f_m) = \nu(f_n) \iff m = n$. So, $\nu(f) = \min_m\{\nu(f_m)\}$.
4) From 3) we have that the semigroup $S^{R_{\mathfrak{m}}}(\nu) = \{\nu(f) \mid 0 \neq f \in R\} = S(\gamma_l \mid l \geqslant 0)$.

Suppose that ν is a rational rank 1 non discrete valuation dominating $R_{\mathfrak{m}}$. We will say that ν has a generating sequence of eigenfunctions for H if all Q_l in the generating sequence (2.2) of Section 2 are eigenfunctions for H.

3. Valuation Semigroups of Invariant Subrings

Theorem 3.1. — *Let $H \leqslant \mathbb{U}_m \times \mathbb{U}_n$ be as in Remark 1.3. Suppose that ν is a rational rank 1 non discrete valuation dominating $R_{\mathfrak{m}}$, where $R = K[X,Y]$, and $\mathfrak{m} = (X,Y)$. Suppose that ν has a generating sequence (2.2)*

$$Q_0 = X, Q_1 = Y, Q_2, \cdots$$

such that each $Q_l \in R$ is an eigenfunction for H. Let notation be as in Section 2. Then denoting $A = R^H$, and defining $\mathfrak{n} = \mathfrak{m} \cap A$ we have

$$S^{A_{\mathfrak{n}}}(\nu) = \left\{ l\gamma_0 + j_1\gamma_1 + \cdots + j_r\gamma_r \;\middle|\; \begin{array}{l} l \in \mathbb{N},\ r \in \mathbb{N},\ 0 \leqslant j_k < \overline{m_k}\ \forall\ k = 1, \cdots, r \\ \alpha^{la}\beta^{b\sum_{k=1}^{r}[j_k d(k)]} = 1 \\ \forall\ b \equiv ax (mod\ t) \end{array} \right\}. \tag{3.1}$$

Proof. — Let $0 \neq f(X,Y) \in R$, with $\deg_Y(f) = d$. By (2.1), $(\alpha^a, \beta^b) \cdot Y^{d(m)} = \beta^{d(m)b} Y^{d(m)}$. Since Q_m is an eigenfunction of H, we conclude that for $m > 0$,

$$(\alpha^a, \beta^b) \cdot Q_m = \beta^{d(m)b} Q_m = \beta^{\deg_Y(Q_m)b} Q_m \,, \ \forall \ (\alpha^a, \beta^b) \in H. \tag{3.2}$$

We also have, $(\alpha^a, \beta^b) \cdot Q_0 = (\alpha^a, \beta^b) \cdot X = \alpha^a X \,, \forall \ (\alpha^a, \beta^b) \in H$. Now f has an expansion of the form 3) of Section 2. So,

$$\begin{aligned}(\alpha^a, \beta^b) \cdot f &= (\alpha^a, \beta^b) \cdot \sum_{m=0}^{d} [(\sum_l b_{l,m} X^l) Q_1^{j_1(m)} \cdots Q_r^{j_r(m)}] \\ &= \sum_{m=0}^{d} [(\sum_l \alpha^{la} b_{l,m} X^l) \beta^{b \sum_{k=1}^{r} [j_k(m) d(k)]} Q_1^{j_1(m)} \cdots Q_r^{j_r(m)}].\end{aligned}$$

Now, $f \in A \iff \alpha^{la} \beta^{b \sum_{k=1}^{r} [j_k(m) d(k)]} = 1, \forall \ b \equiv ax (\text{mod t}), \forall \ l$, such that $b_{l,m} \neq 0$.
So,

$$\begin{aligned}&\{\nu(f) \,|\, 0 \neq f \in A_{\mathfrak{n}}\} = \{\nu(f) \,|\, 0 \neq f \in A\} \\ &\subset \left\{ l\gamma_0 + j_1\gamma_1 + \cdots + j_r\gamma_r \;\middle|\; \begin{array}{l} l \in \mathbb{N},\, r \in \mathbb{N},\, 0 \leqslant j_k < \overline{m_k} \ \forall \ k = 1, \cdots, r \\ \alpha^{la} \beta^{b \sum_{k=1}^{r} [j_k d(k)]} = 1 \\ \forall \ b \equiv ax (\text{mod t}) \end{array} \right\}.\end{aligned}$$

Conversely, suppose we have $l \in \mathbb{N}$, $r \in \mathbb{N}$, $0 \leqslant j_k < \overline{m_k} \ \forall \ k = 1, \cdots, r$ such that $\forall \ b \equiv ax (\text{mod t})$ we have $\alpha^{la} \beta^{b \sum_{k=1}^{r} [j_k d(k)]} = 1$. Define $f(X,Y) = X^l Q_1^{j_1} \cdots Q_r^{j_r} \in R$. For any $(\alpha^a, \beta^b) \in H$ we have, $(\alpha^a, \beta^b) \cdot f = (\alpha^a, \beta^b) \cdot (X^l Q_1^{j_1} \cdots Q_r^{j_r}) = \alpha^{la} \beta^{b \sum_{k=1}^{r} [j_k d(k)]} X^l Q_1^{j_1} \cdots Q_r^{j_r} = f$, that is, $f \in A$. So $\nu(f) = l\gamma_0 + j_1\gamma_1 + \cdots + j_r\gamma_r \in S^{A_{\mathfrak{n}}}(\nu)$. Hence we conclude,

$$S^{A_{\mathfrak{n}}}(\nu) = \left\{ l\gamma_0 + j_1\gamma_1 + \cdots + j_r\gamma_r \;\middle|\; \begin{array}{l} l \in \mathbb{N},\, r \in \mathbb{N},\, 0 \leqslant j_k < \overline{m_k} \ \forall \ k = 1, \cdots, r \\ \alpha^{la} \beta^{b \sum_{k=1}^{r} [j_k d(k)]} = 1 \\ \forall \ b \equiv ax (\text{mod t}) \end{array} \right\}.$$

□

4. Finite and Non-Finite Generation

In this section we study the finite and non-finite generation of the valuation semigroup $S^{R_{\mathfrak{m}}}(\nu)$ over the subsemigroup $S^{A_{\mathfrak{n}}}(\nu)$. A semigroup S is said to be finitely generated over a subsemigroup T if there are finitely many elements $s_1, \cdots, s_n$ in S such that $S = \{s_1, \cdots, s_n\} + T$.

At the end of this section we will prove the following theorem.

Theorem 4.1. — *Let $R_{\mathfrak{m}} = K[X,Y]_{(X,Y)}$ and $H \leqslant \mathbb{U}_m \times \mathbb{U}_n$ be as in Remark 1.3.*

1) *$\exists$ a rational rank 1 non discrete valuation ν dominating $R_{\mathfrak{m}}$ with a generating sequence (2.2) of eigenfunctions for $H \iff (m,n) = t$.*
2) *If $(m,n) = t = 1$, then $S^{R_{\mathfrak{m}}}(\nu)$ is a finitely generated $S^{A_{\mathfrak{n}}}(\nu)$-module for all rational rank 1 non discrete valuations ν which dominate $R_{\mathfrak{m}}$ and have a generating sequence (2.2) of eigenfunctions for H.*
3) *If $(m,n) = t > 1$, then $S^{R_{\mathfrak{m}}}(\nu)$ is not a finitely generated $S^{A_{\mathfrak{n}}}(\nu)$-module for all rational rank 1 non discrete valuations ν which dominate $R_{\mathfrak{m}}$ and have a generating sequence (2.2) of eigenfunctions for H.*

We introduce some notation. Let $\sigma(0) = 0$, $\sigma(l) = \min\{j \,|\, j > \sigma(l-1) \text{ and } \overline{m_j} > 1\}$. Let $P_l = Q_{\sigma(l)}$ and $\beta_l = \nu(P_l) = \gamma_{\sigma(l)}\ \forall\ l \geqslant 0$. Let $\overline{n_l} = [G(\beta_0, \cdots, \beta_l) : G(\beta_0, \cdots, \beta_{l-1})] = \min\{q \in \mathbb{Z}_{>0} \,|\, q\beta_l \in G(\beta_0, \cdots, \beta_{l-1})\}\ \forall\ l \geqslant 1$. Then $\overline{n_l} = \overline{m_{\sigma(l)}}$. $S^{R_{\mathfrak{m}}}(\nu) = S(\gamma_0, \gamma_1, \cdots) = S(\beta_0, \beta_1, \cdots)$ and $\{\beta_l\}_{l \geqslant 0}$ form a minimal generating set of $S^{R_{\mathfrak{m}}}(\nu)$, that is, $\overline{n_l} > 1\ \forall\ l \geqslant 1$.

We first make a general observation. Suppose for some $d \geqslant 1$, $j_r \neq 0$ and $l, j_1, \cdots, j_r \in \mathbb{N}$, we have an expression of the form, $\beta_d = l\beta_0 + j_1\beta_1 + \cdots + j_r\beta_r$. If $r > d$ then $j_r\beta_r \geqslant \beta_r > \beta_d$ which is a contradiction. If $r < d$ then $\beta_d \in G(\beta_0, \cdots, \beta_{d-1}) \implies \overline{n_d} = 1$. This is a contradiction as $\overline{n_l} > 1\ \forall\ l \geqslant 1$. Thus, $\beta_r = l\beta_0 + j_1\beta_1 + \cdots + j_r\beta_r$. If $j_r > 1$, then $j_r\beta_r > \beta_r$. If $j_r = 0$, then $\beta_r \in G(\beta_0, \cdots, \beta_{r-1}) \implies \overline{n_r} = 1$. So, $j_r = 1$. Since $\beta_i > 0\ \forall\ i$, we then have $l = 0, j_i = 0\ \forall\ i \neq r$. Thus, for $l, j_1, \cdots, j_r \in \mathbb{N}$ and $d \geqslant 1$,

$$\beta_d = l\beta_0 + j_1\beta_1 + \cdots + j_r\beta_r \implies j_d = 1, l = 0, j_i = 0\ \forall\ i \neq d. \tag{4.1}$$

Proposition 4.2. — *Let $R_{\mathfrak{m}} = K[X,Y]_{(X,Y)}$ and $H \leqslant \mathbb{U}_m \times \mathbb{U}_n$ be as in Remark 1.3. Let assumptions be as in Theorem 3.1. Then $S^{R_{\mathfrak{m}}}(\nu)$ is finitely generated over the subsemigroup $S^{A_{\mathfrak{n}}}(\nu)$ if and only if $\exists\ N \in \mathbb{Z}_{>0}$ such that $Q_r \in A\ \forall\ r \geqslant N$. Further, if $Q_N \in A$, then $Q_M \in A\ \forall\ M \geqslant N \geqslant 1$.*

Proof. — We first show that, for any $r \geqslant 1$, $\gamma_r \in S^{A_{\mathfrak{n}}}(\nu) \iff Q_r \in A$. It is enough to show the implication $\gamma_r \in S^{A_{\mathfrak{n}}}(\nu) \implies Q_r \in A$. From (3.1) we have, $\gamma_r \in S^{A_{\mathfrak{n}}}(\nu) \implies \gamma_r = l\gamma_0 + j_1\gamma_1 + \cdots + j_s\gamma_s$, where $l \in \mathbb{N}$, $s \in \mathbb{N}$, $0 \leqslant j_k < \overline{m_k}$ and $\alpha^{la}\beta^{b\sum_{k=1}^{s} j_k d(k)} = 1\ \forall\ b \equiv ax (\mathrm{mod}\ t)$.
Since $l, j_1, \cdots, j_s \in \mathbb{N}$, $\gamma_i < \gamma_{i+1}\ \forall\ i \geqslant 1$ and $\gamma_i > 0\ \forall\ i$, we have $r \geqslant s$. If $r = s$, then $\gamma_r = l\gamma_0 + \sum_{k=1}^{r} j_k\gamma_k \geqslant j_r\gamma_r \geqslant \gamma_r$. Since $j_r \neq 0$ and $j_r \in \mathbb{N}$ we have $j_r = 1$. And $\gamma_i > 0\ \forall\ i$ implies $l = j_1 = \cdots = j_{r-1} = 0$. Then $\beta^{bd(r)} = 1\ \forall\ b \equiv ax (\mathrm{mod}\ t)$. So from (3.2), $(\alpha^a, \beta^b) \cdot Q_r = Q_r\ \forall\ b \equiv ax (\mathrm{mod}\ t)$, that is, $Q_r \in A$.

If $r > s$, then $\gamma_r = l\gamma_0 + \sum_{k=1}^{s} j_k\gamma_k \implies \overline{m_r} = 1$. Since $0 \leqslant j_k < \overline{m_k}$, by Equation (8) in [[6]] we have $Q_{r+1} = Q_r - \lambda X^l Y^{j_1} Q_2^{j_2} \cdots Q_s^{j_s}$ where $\lambda \in K \setminus \{0\}$. Since each Q_m is an eigenfunction for H, from (3.2) we have, $\forall\ b \equiv ax (\mathrm{mod}\ t)$,

$$\beta^{bd(r+1)} Q_{r+1} = \beta^{bd(r)} Q_r - \lambda \alpha^{la} \beta^{b\sum_{k=1}^{s} j_k d(k)} X^l Y^{j_1} Q_2^{j_2} \cdots Q_s^{j_s}.$$

Again by 2) in Section 2 we have $d(r+1) = \overline{m_1} \cdots \overline{m_r} = \overline{m_1} \cdots \overline{m_{r-1}} = d(r)$, as $\overline{m_r} = 1$. So the above expression yields $\beta^{bd(r)} Q_{r+1} = \beta^{bd(r)} Q_r - \lambda \alpha^{la} \beta^{b\sum_{k=1}^{s} j_k d(k)} X^l Y^{j_1} Q_2^{j_2} \cdots Q_s^{j_s}$ $\forall\ b \equiv ax (\mathrm{mod}\ t)$. Since Q_{r+1} is an eigenfunction, this implies $\beta^{bd(r)} = \alpha^{la} \beta^{b\sum_{k=1}^{s} j_k d(k)} = 1\ \forall\ b \equiv ax (\mathrm{mod}\ t)$. From (3.2), we then have $Q_r \in A$.

To prove the proposition, we now show $S^{R_{\mathfrak{m}}}(\nu)$ is finitely generated over the subsemigroup $S^{A_{\mathfrak{n}}}(\nu)$ if and only if $\exists\ N \in \mathbb{Z}_{>0}$ such that $\forall\ r \geqslant N$, $\gamma_r \in S^{A_{\mathfrak{n}}}(\nu)$.
Suppose $S^{R_{\mathfrak{m}}}(\nu)$ is finitely generated over $S^{A_{\mathfrak{n}}}(\nu)$. So, $\exists\ x_0, \cdots, x_l \in S^{R_{\mathfrak{m}}}(\nu)$ such that $S^{R_{\mathfrak{m}}}(\nu) = \{x_0, \cdots, x_l\} + S^{A_{\mathfrak{n}}}(\nu)$. Let $L \in \mathbb{N}$ be the least natural number such that $S^{R_{\mathfrak{m}}}(\nu) = S(\beta_0, \cdots, \beta_L) + S^{A_{\mathfrak{n}}}(\nu)$, where $\beta_i = \gamma_{\sigma(i)}\ \forall\ i \geqslant 0$. Let $M > L$. Now β_M has an expression $\beta_M = \sum_{i=0}^{L} a_i\beta_i + y$ where $y \in S^{A_{\mathfrak{n}}}(\nu)$, $a_i \in \mathbb{N}$. From (3.1) we have $\beta_M = \sum_{i=0}^{L} a_i\beta_i + (l\gamma_0 + j_1\gamma_1 + \cdots + j_s\gamma_s)$, where $0 \leqslant j_k < \overline{m_k}$ and $\alpha^{la}\beta^{b\sum_{k=1}^{s} j_k d(k)} = 1\ \forall\ b \equiv ax (\mathrm{mod}\ t)$. We observe $\overline{m_k} = 1 \implies j_k = 0$. Thus the above expression can be rewritten as,

$$\beta_M = \sum_{i=0}^{L} a_i\beta_i + (l\beta_0 + j_1\beta_1 + \cdots + j_p\beta_p)$$

where $0 \leqslant j_k < \overline{n_k}$ and $\alpha^{la}\beta^{b\sum_{k=1}^{p} j_k \deg_Y(P_k)} = 1\ \forall\ b \equiv ax (\mathrm{mod}\ t)$. Since $L < M$, from (4.1) we obtain $j_M = 1, a_i = 0\ \forall\ i = 0, \cdots, L$ and $j_k = 0\ \forall\ k \neq M$. Thus $\beta^{b\deg_Y(P_M)} = 1\ \forall\ b \equiv ax (\mathrm{mod}\ t) \implies n \mid \deg_Y(P_M)$. Thus $n \mid d(\sigma(M))\ \forall\ M > L$. From 2) in Section 2 we have $n \mid d(r)\ \forall\ r \geqslant \sigma(L+1)$. So, $\beta^{bd(r)} = 1\ \forall\ b \equiv ax (\mathrm{mod}\ t)$. From (3.2) we conclude, $Q_r \in A\ \forall\ r \geqslant \sigma(L+1)$, that is, $\gamma_r \in S^{A_{\mathfrak{n}}}(\nu)\ \forall\ r \geqslant \sigma(L+1)$.

Conversely, we assume $S(\gamma_N, \gamma_{N+1}, \cdots) \subset S^{A_{\mathfrak{n}}}(\nu)$ for some $N \in \mathbb{Z}_{>0}$. Now $\gamma_i \in \mathbb{Q}_{>0}\ \forall\ i$ implies $\forall\ i \neq j$, $\exists\ d_i, d_j \in \mathbb{Z}_{>0}$ such that $d_i\gamma_i = d_j\gamma_j$. We thus have $d_i\gamma_i = d_{i,N}\gamma_N\ \forall\ i = 0, \cdots, N-1$. We will now show that, $S^{R_{\mathfrak{m}}}(\nu) = T + S^{A_{\mathfrak{n}}}(\nu)$, where $T = \{\sum_{i=0}^{N-1} \overline{a_i}\gamma_i \mid 0 \leqslant \overline{a_i} < d_i\}$. Now, $\gamma_i \in S^{R_{\mathfrak{m}}}(\nu)\ \forall\ i = 0, \cdots, N-1 \implies T + S^{A_{\mathfrak{n}}}(\nu) \subset S^{R_{\mathfrak{m}}}(\nu)$. So it is enough to show $S^{R_\mathfrak{m}}(\nu) \subset T + S^{A_\mathfrak{n}}(\nu)$.

$$\begin{aligned} x \in S^{R_\mathfrak{m}}(\nu) \implies x &= \sum_{i=0}^{N-1} a_i\gamma_i + \sum_{i=N}^{l} a_i\gamma_i \\ \implies x &= \sum_{i=0}^{N-1} \overline{a_i}\gamma_i + \sum_{i=0}^{N-1} b_i d_i\gamma_i + \sum_{i=N}^{l} a_i\gamma_i \\ &\qquad \text{where } a_i = \overline{a_i} + b_i d_i,\ 0 \leqslant \overline{a_i} < d_i,\ b_i \in \mathbb{N} \\ \implies x &= \sum_{i=0}^{N-1} \overline{a_i}\gamma_i + \sum_{i=0}^{N-1} b_i d_{i,N}\gamma_N + \sum_{i=N}^{l} a_i\gamma_i \\ \implies x &= \sum_{i=0}^{N-1} \overline{a_i}\gamma_i + y,\ \text{ where } y \in S^{A_\mathfrak{n}}(\nu). \end{aligned}$$

Thus we have shown $S^{R_\mathfrak{m}}(\nu) \subset T + S^{A_\mathfrak{n}}(\nu)$. Since T is a finite set, we have $S^{R_\mathfrak{m}}(\nu)$ is finitely generated over $S^{A_\mathfrak{n}}(\nu)$.

From (3.2), $(\alpha^a, \beta^b) \cdot Q_N = \beta^{d(N)b} Q_N \ \forall \ b \equiv ax(\text{mod } t)$. So, $Q_N \in A \iff \beta^{d(N)b} = 1 \ \forall \ b \equiv ax(\text{mod } t)$. Again from 2) of Section 2 we have $d(N) \mid d(M) \ \forall \ M \geqslant N \geqslant 1$. Hence we obtain, $Q_N \in A \implies Q_M \in A \ \forall \ M \geqslant N \geqslant 1$. So, $S^{R_\mathfrak{m}}(\nu)$ is not finitely generated over $S^{A_\mathfrak{n}}(\nu)$ if and only if $Q_r \notin A \ \forall \ r \geqslant 1$. □

Lemma 4.3. — *Let $H \leqslant \mathbb{U}_m \times \mathbb{U}_n$ be as in Remark 1.3. Let assumptions be as in Theorem 3.1. Then $S^{R_\mathfrak{m}}(\nu)$ is not finitely generated over $S^{A_\mathfrak{n}}(\nu)$ if and only if $n \nmid d(l) \ \forall \ l \geqslant 2$.*

Proof. — Suppose that $S^{R_\mathfrak{m}}(\nu)$ is not finitely generated over $S^{A_\mathfrak{n}}(\nu)$. Then $Q_l \notin A \ \forall \ l \geqslant 1$. From (3.2), if $n \mid d(l)$, then, $(\alpha^a, \beta^b) \cdot Q_l = \beta^{d(l)b} Q_l = Q_l$, that is $Q_l \in A$, which is a contradiction. So, $n \nmid d(l) \ \forall \ l \geqslant 2$.

Conversely, suppose $n \nmid d(l) \ \forall \ l \geqslant 2$, that is, $n \nmid d(l) \ \forall \ l \geqslant 1$. Now, $(x, t) = 1 \implies ax \equiv 1(\text{mod } t)$ for some $a \in \mathbb{Z}$, so, $(\alpha^a, \beta) \in H$. From (3.2), $(\alpha^a, \beta) \cdot Q_l = \beta^{d(l)} Q_l \neq Q_l$ for all $l \geqslant 1$, as $n \nmid d(l)$. So we have $Q_l \notin A \ \forall \ l \geqslant 1$. Hence $S^{R_\mathfrak{m}}(\nu)$ is not finitely generated over $S^{A_\mathfrak{n}}(\nu)$. □

Proposition 4.4. — *Let $H \leqslant \mathbb{U}_m \times \mathbb{U}_n$ be as in Remark 1.3, such that $(m, n) > t \geqslant 1$. Suppose that ν is a rational rank 1 non discrete valuation dominating $R_\mathfrak{m}$, with a generating sequence (2.2) $\{Q_l\}_{l \geqslant 0}$, where $Q_0 = X, Q_1 = Y$ as in Section 2. Then $\{Q_l\}_{l \geqslant 0}$ is not a sequence of eigenfunctions for H.*

Proof. — Let $d = (m, n)$. Then $1 \leqslant t < d \leqslant \min\{m, n\}$. So, $t < m$ and $t < n$. We recall, $H = \{(\alpha^a, \beta^b) \mid b \equiv ax(\text{mod } t)\}$. Thus $(\alpha^t, 1)$, $(1, \beta^t) \in H$. Let $\{Q_l\}_{l \geqslant 0}$ be the generating sequence (2.2) with $Q_0 = X$, $Q_1 = Y$.

Let $\nu(Q_l) = \gamma_l \ \forall \ l \geqslant 0$. By Equation (8) in [[6]], $Q_2 = Y^s - \lambda X^r$, where $\lambda \in K \setminus \{0\}$, $s\gamma_1 = r\gamma_0$, and $s = \min\{q \in \mathbb{Z}_{>0} \mid q\gamma_1 \in \gamma_0\mathbb{Z}\}$. From (2.1), we have,

$$(\alpha^t, 1) \cdot Q_2 = (\alpha^t, 1) \cdot [Y^s - \lambda X^r] = Y^s - \lambda\alpha^{rt}X^r.$$
$$(1, \beta^t) \cdot Q_2 = (1, \beta^t) \cdot [Y^s - \lambda X^r] = \beta^{st}Y^s - \lambda X^r.$$

If Q_2 was an eigenfunction of H, then $m \mid rt \implies r = r_1\frac{m}{t}$, where $r_1 \in \mathbb{Z}_{>0}$. Similarly, $n \mid st \implies s = s_1\frac{n}{t}$, where $s_1 \in \mathbb{Z}_{>0}$. And, $s\gamma_1 = r\gamma_0 \implies s_1\frac{n}{t}\gamma_1 = r_1\frac{m}{t}\gamma_0$. So, $s_1\frac{n}{d}\gamma_1 = r_1\frac{m}{d}\gamma_0$. Now, $d \,|\, n$ implies $s_1\frac{n}{d} \in \mathbb{Z}_{>0}$. Similarly, $r_1\frac{m}{d} \in \mathbb{Z}_{>0}$. Thus, $s_1\frac{n}{d}\gamma_1 \in \gamma_0\mathbb{Z}$. But $t < d$ implies $s_1\frac{n}{d} < s_1\frac{n}{t} = s$, and this contradicts the minimality of s. Thus Q_2 is not an eigenfunction of H. So, $\{Q_l\}_{l\geqslant 0}$ is not a generating sequence of eigenfunctions for H. □

We know, if ω is a primitive l-th root of unity in K, then $\{\omega^k \mid 1 \leqslant k \leqslant l\}$ is a complete list of all l-th roots of unity in K, and $\{\omega^k \mid 1 \leqslant k \leqslant l \text{ and } (k,l) = 1\}$ is a complete list of all primitive l-th roots of unity in K. We have, α is a primitive m-th root of unity and β is a primitive n-th root of unity in K. Let δ be a primitive mn-th root of unity in K. Then δ^n is a primitive m-th root of unity. Now, $S_\alpha = \{\alpha^k \mid 1 \leqslant k \leqslant m \text{ and } (k,m) = 1\}$ is a complete list of all primitive m-th roots of unity in K. And, $S_{\delta^n} = \{\delta^{kn} \mid 1 \leqslant k \leqslant m \text{ and } (k,m) = 1\}$ is also a complete list of all primitive m-th roots of unity. Thus, $\alpha = \delta^{w_1 n}$ where $(w_1, m) = 1$ and $1 \leqslant w_1 \leqslant m$. Similarly, $\beta = \delta^{w_2 m}$ where $(w_2, n) = 1$ and $1 \leqslant w_2 \leqslant n$.

Remark 4.5. — Let $p, q \in \mathbb{Z}$. With the notation introduced above, $\beta^p = \alpha^q \iff \frac{pw_2}{n} - \frac{qw_1}{m} \in \mathbb{Z}$.

Proof. — We have, $\beta = \delta^{w_2 m}$ and $\alpha = \delta^{w_1 n}$, where δ is a primitive mn-th root of unity.
Thus, $\beta^p = \alpha^q \iff \delta^{w_2 mp} = \delta^{w_1 nq} \iff mn \mid (w_2mp - w_1nq) \iff \frac{pw_2}{n} - \frac{qw_1}{m} \in \mathbb{Z}$. □

Proposition 4.6. — *Let $H \leqslant \mathbb{U}_m \times \mathbb{U}_n$ be as in Remark 1.3, such that $(m,n) = t$, $t > 1$. Set $m = Mt$, and $n = Nt$, where $M, N \in \mathbb{Z}_{>0}$ and $(M,N) = 1$. Suppose that $\exists$ a prime number p such that $p \mid t$ but $p \nmid N$. Suppose that ν is a rational rank 1 non discrete valuation dominating $R_\mathfrak{m}$ with a generating sequence (2.2) of eigenfunctions for H. Then $S^{R_\mathfrak{m}}(\nu)$ is not finitely generated over $S^{A_\mathfrak{n}}(\nu)$.*

Proof. — Let $\{Q_l\}_{l\geqslant 0}$ be the generating sequence (2.2) of the valuation ν, where $Q_0 = X, Q_1 = Y$, and each Q_l is an eigenfunction for H. Let $\gamma_l = \nu(Q_l) \ \forall \ l \geqslant 0$. Without any loss of generality, we can assume $\gamma_0 = 1$. Since ν is a rational valuation, we can write $\gamma_k = \frac{a_k}{b_k} \ \forall \ k \geqslant 1$, where $(a_k, b_k) = 1$. We have, $p \mid t$, and $p \nmid N$ for a prime p. So $(p, N) = 1$. So $\exists \ N_1 \in \mathbb{Z}$ such that $NN_1 \equiv 1 (\text{mod } p)$. Let w_1 and w_2 be as in Remark 4.5.

Now $(m, w_1) = 1$ and $t \mid m$. So $(t, w_1) = 1$. So $(p, w_1) = 1$. So $\exists\ \overline{w_1} \in \mathbb{Z}$ such that $w_1\overline{w_1} \equiv 1 (\text{mod } p)$.
We now use induction to show the following $\forall\ k \geqslant 1$,

$$\begin{aligned} (p, \overline{m_k}) = 1,\ (p, b_k) = 1 \\ a_k \equiv b_k M N_1 x w_2 \overline{w_1} d(k)\,(\text{mod } p). \end{aligned} \tag{4.2}$$

We have $\gamma_1 = \frac{a_1}{b_1}$, where $(a_1, b_1) = 1$. So $\overline{m_1} = b_1$. By Equation (8) in [[6]], we have $Q_2 = Y^{b_1} - \lambda_1 X^{a_1}$, for some $\lambda_1 \in K \setminus \{0\}$. Again $(\alpha, \beta^x) \in H$. Now, $(\alpha, \beta^x) \cdot Q_2 = \beta^{b_1 x} Y^{b_1} - \lambda_1 \alpha^{a_1} X^{a_1}$. Since Q_2 is an eigenfunction for H, we have

$$\begin{aligned} \beta^{b_1 x} = \alpha^{a_1} &\Longrightarrow \frac{b_1 x w_2}{n} - \frac{a_1 w_1}{m} \in \mathbb{Z} \text{ by Remark 4.5} \\ &\Longrightarrow \frac{b_1 x w_2}{Nt} - \frac{a_1 w_1}{Mt} \in \mathbb{Z} \\ &\Longrightarrow MNt \mid [b_1 x M w_2 - a_1 N w_1] \\ &\Longrightarrow b_1 M N_1 x w_2 \overline{w_1} \equiv a_1 (\text{mod } p) \text{ as } p \mid t. \end{aligned}$$

If $(p, b_1) \neq 1$, then $p \mid b_1 \Longrightarrow p \mid a_1$. But this contradicts $(a_1, b_1) = 1$. So, $(p, b_1) = 1$. Since $\overline{m_1} = b_1$, we thus have $(p, \overline{m_1}) = 1$. Thus we have the induction step for $k = 1$.
Suppose (4.2) is true for $k = 1, \cdots, l-1$. From (3.2) we have $(\alpha^a, \beta^b) \cdot Q_k = \beta^{d(k)b} Q_k\ \ \forall\ k \geqslant 1, \forall\ (\alpha^a, \beta^b) \in H$. By Equation (8) in [[6]] we have, $Q_{l+1} = Q_l^{\overline{m_l}} - \lambda_l X^{c_0} Y^{c_1} Q_2^{c_2} \cdots Q_{l-1}^{c_{l-1}}$ where $\lambda_l \in K \setminus \{0\}$, $0 \leqslant c_k < \overline{m_k}\ \ \forall\ k = 1, \cdots, l-1$ and $\overline{m_l}\gamma_l = \sum_{k=0}^{l-1} c_k \gamma_k$.
$(\alpha, \beta^x) \cdot Q_{l+1} = \beta^{x\overline{m_l}d(l)} Q_l^{\overline{m_l}} - \lambda_l \alpha^{c_0} \beta^{x[\sum_{k=1}^{l-1} c_k d(k)]} X^{c_0} Y^{c_1} Q_2^{c_2} \cdots Q_{l-1}^{c_{l-1}}$. Since Q_{l+1} is an eigenfunction for H, we have

$$\begin{aligned} &\beta^{x\overline{m_l}d(l)} = \alpha^{c_0} \beta^{x[\sum_{k=1}^{l-1} c_k d(k)]} \\ \Longrightarrow &\beta^{x[\overline{m_l}d(l) - \sum_{k=1}^{l-1} c_k d(k)]} = \alpha^{c_0} \\ \Longrightarrow &\frac{x[\overline{m_l}d(l) - \sum_{k=1}^{l-1} c_k d(k)] w_2}{Nt} - \frac{c_0 w_1}{Mt} \in \mathbb{Z} \text{ by Remark 4.5} \\ \Longrightarrow &MNt \mid \left[M x w_2 \overline{m_l} d(l) - M x w_2 \sum_{k=1}^{l-1} c_k d(k) - N c_0 w_1 \right] \\ \Longrightarrow &p \mid \left[M x w_2 \overline{m_l} d(l) - M x w_2 \sum_{k=1}^{l-1} c_k d(k) - N c_0 w_1 \right] \\ \Longrightarrow &M N_1 x w_2 \overline{w_1}\, \overline{m_l} d(l) \equiv \left[M N_1 x w_2 \overline{w_1} \sum_{k=1}^{l-1} c_k d(k) + c_0 \right] (\text{mod } p). \end{aligned}$$

Now, $p \mid \overline{m_l} \implies c_0 = \lambda p - MN_1xw_2\overline{w_1}\sum_{k=1}^{l-1} c_k d(k)$, where $\lambda \in \mathbb{Z}$. Let $\overline{m_l} = pM_l$, where $M_l \in \mathbb{Z}_{>0}$. So, $\overline{m_l}\gamma_l = pM_l\gamma_1 = c_0 + \sum_{k=1}^{l-1} c_k\gamma_k = \lambda p + \sum_{k=1}^{l-1} c_k[\gamma_k - MN_1xw_2\overline{w_1}d(k)]$.
By our induction statement, $\forall\ k = 1, \cdots, l-1$, we have $a_k = t_k p + b_k MN_1xw_2\overline{w_1}d(k)$, where $t_k \in \mathbb{Z}$. Thus,

$$\begin{aligned} pM_l\gamma_l &= \lambda p + \sum_{k=1}^{l-1} c_k[\frac{t_kp + b_kMN_1xw_2\overline{w_1}d(k)}{b_k} - MN_1xw_2\overline{w_1}d(k)] \\ &= \lambda p + p\sum_{k=1}^{l-1} c_k t_k \frac{1}{b_k}. \end{aligned}$$

Now $(a_k, b_k) = 1 \Longrightarrow \exists\ h_k \in \mathbb{Z}$ such that $h_k a_k \equiv 1(\text{mod } b_k)$. Let $h_k a_k - 1 = \zeta_k b_k$, where $\zeta_k \in \mathbb{Z}$. So, $\frac{1}{b_k} = \frac{h_k a_k - (h_k a_k - 1)}{b_k} = h_k\gamma_k - \zeta_k$. Then, $pM_l\gamma_l = \lambda p + p\sum_{k=1}^{l-1} c_k t_k[h_k\gamma_k - \zeta_k]$ implies

$$M_l\gamma_l = \lambda + \sum_{k=1}^{l-1} c_k t_k [h_k\gamma_k - \zeta_k] \in G(\gamma_0, \cdots, \gamma_{l-1}).$$

But this contradicts the minimality of $\overline{m_l}$. So $p \nmid \overline{m_l}$. So $(p, \overline{m_l}) = 1$.

Now, $\overline{m_l}\gamma_l = c_0 + \sum_{k=1}^{l-1} c_k\gamma_k \implies \overline{m_l}\frac{a_l}{b_l} = c_0 + \sum_{k=1}^{l-1} c_k\frac{a_k}{b_k} \implies \overline{m_l}a_l\prod_{k=1}^{l-1} b_k = c_0B + B\sum_{k=1}^{l-1} c_k\frac{a_k}{b_k}$, where $B = \prod_{k=1}^{l} b_k$. From the induction hypothesis, $\frac{a_k}{b_k}B = [t_kp + b_kMN_1xw_2\overline{w_1}d(k)]\frac{B}{b_k}$. So,

$$\begin{aligned} \overline{m_l}a_l\prod_{k=1}^{l-1} b_k &= c_0B + \sum_{k=1}^{l-1} c_k[t_kp + b_kMN_1xw_2\overline{w_1}d(k)]\frac{B}{b_k} \\ \Longrightarrow \overline{m_l}a_l\prod_{k=1}^{l-1} b_k &\equiv [c_0 + MN_1xw_2\overline{w_1}\sum_{k=1}^{l-1} c_k d(k)]B(\text{mod } p). \end{aligned}$$

Since, $MN_1xw_2\overline{w_1}\,\overline{m_l}d(l) \equiv [MN_1xw_2\overline{w_1}\sum_{k=1}^{l-1} c_kd(k) + c_0](\text{mod } p)$, we have

$$\overline{m_l}a_l\prod_{k=1}^{l-1} b_k \equiv MN_1xw_2\overline{w_1}\,\overline{m_l}d(l)\prod_{k=1}^{l} b_k(\text{mod } p).$$

Since $(p, \overline{m_l}) = 1$, $(p, b_k) = 1\ \ \forall\ \ k = 1, \cdots, l-1$, we have $a_l \equiv MN_1xw_2\overline{w_1}d(l)b_l\,(\text{mod } p)$. If $p \mid b_l$, then $p \mid a_l$ which contradicts $(a_l, b_l) = 1$. So $(p, b_l) = 1$. Thus we have the induction step for $k = l$.

In particular, by induction we have $(p, \overline{m_k}) = 1\ \forall\ k \geqslant 1$. Since $d(k) = \overline{m_1}\cdots\overline{m_{k-1}}$ (by 2), Section 2), we have $(p, d(k)) = 1\ \forall\ k \geqslant 2$. So $p \nmid d(k)\ \forall\ k \geqslant 2 \Longrightarrow t \nmid d(k)\ \forall\ k \geqslant 2 \Longrightarrow n = Nt \nmid d(k)\ \forall\ k \geqslant 2$. Thus by Lemma 4.3, we have $S^{R_\mathfrak{m}}(\nu)$ is not finitely generated over $S^{A_\mathfrak{n}}(\nu)$. $\square$

Proposition 4.7. — *Let $H \leqslant \mathbb{U}_m \times \mathbb{U}_n$ be as in Remark 1.3, such that $(m,n) = t$ and $t > 1$. Set $m = Mt$ and $n = Nt$ where $M, N \in \mathbb{Z}_{>0}$ and $(M,N) = 1$. Suppose that for any prime number p which divides t, the number p also divides N. Suppose that ν is a rational rank 1 non discrete valuation dominating $R_{\mathfrak{m}}$ with a generating sequence (2.2) of eigenfunctions for H. Then $S^{R_{\mathfrak{m}}}(\nu)$ is not finitely generated over $S^{A_{\mathfrak{n}}}(\nu)$.*

Proof. — Since $(x,t) = 1$, $\exists\ r \in \mathbb{Z}_{>0}$ such that $rx \equiv 1 (\text{mod } t)$. So $(r,t) = 1$. Recall, $\alpha = \delta^{w_1 n}, \beta = \delta^{w_2 m}$, where δ is a primitive mn-th root of unity, and $(w_1, m) = 1, (w_2, n) = 1, 1 \leqslant w_1 \leqslant m$ and $1 \leqslant w_2 \leqslant n$. Now, $M \mid m \implies (w_1, M) = 1$. Similarly, $(w_2, N) = 1$, $(w_1, t) = 1$, $(w_2, t) = 1$. So $\exists\ \overline{w_1}, \overline{w_2} \in \mathbb{Z}_{>0}$ such that $w_1\overline{w_1} \equiv 1 (\text{mod } t)$ and $w_2\overline{w_2} \equiv 1 (\text{mod } t)$.
Write $N = \overline{N}N'$, where $\overline{N}$ is the largest factor of N such that $(\overline{N}, x) = 1$. If $\overline{N} = 1$, then for any prime p dividing N, we have $p \mid x$. So in particular $p \mid t \implies p \mid x$. But this is a contradiction as $(t,x) = 1$. So $\overline{N} > 1$ if $N > 1$. We will now show $(\overline{N}, N') = 1$. Suppose the contrary. Then $\exists$ a prime p such that $p \mid \overline{N}$ and $p \mid N'$. $p \mid \overline{N} \implies (p,x) = 1 \implies (\overline{N}p, x) = 1$. And, $\overline{N}N' = N \implies p\overline{N} \mid N$. This contradicts the maximality of $\overline{N}$. So $(\overline{N}, N') = 1$. Hence $(N,x) = (N',x)$. We will now show that $(t, N') = 1$. Suppose $\exists$ a prime p such that $p \mid t$ and $p \mid N'$. Then $p \mid t, p \mid N$ and $p \nmid \overline{N}$. Thus $p \mid t$ and $p \mid x$, which is a contradiction as t and x are coprime. Thus $(t, N') = 1$. Also $(N, w_2) = 1$ implies $(\overline{N}, w_2) = 1$.
Let $\{Q_l\}_{l \geqslant 0}$ be the generating sequence (2.2) of the valuation ν, where $Q_0 = X, Q_1 = Y$, and each Q_l is an eigenfunction for H. Let $\gamma_l = \nu(Q_l)\ \forall\ l \geqslant 0$. Without any loss of generality, we can assume $\gamma_0 = 1$. Let $\gamma_1 = \frac{a_1}{b_1}$, where $(a_1, b_1) = 1$. So $\overline{m_1} = b_1$. By Equation (8) in [[6]], we have $Q_2 = Y^{b_1} - \zeta_1 X^{a_1}$ for some $\zeta_1 \in K \setminus \{0\}$. Now,$(\alpha, \beta^x) \in H$. By (3.2), $(\alpha^a, \beta^b) \cdot Q_k = \beta^{d(k)b} Q_k\ \forall\ k \geqslant 1$, $\forall\ (\alpha^a, \beta^b) \in H$. So, $(\alpha, \beta^x) \cdot Q_2 = (\alpha, \beta^x) \cdot [Y^{b_1} - \zeta_1 X^{a_1}] = \beta^{b_1 x} Y^{b_1} - \zeta_1 \alpha^{a_1} X^{a_1}$. Since Q_2 is an eigenfunction for H, we have

$$\begin{aligned} \beta^{b_1 x} = \alpha^{a_1} &\implies \frac{b_1 x w_2}{Nt} - \frac{a_1 w_1}{Mt} \in \mathbb{Z} \text{ by Remark 4.5} \\ &\implies M\overline{N}t \mid [Mb_1 x w_2 - N a_1 w_1] \\ &\implies M \mid a_1 \text{ and } \overline{N} \mid b_1 \text{ as } (\overline{N}, w_2) = 1,\ (M, w_1) = 1, \\ &\qquad\qquad (M,N) = 1,\ (\overline{N}, x) = 1. \end{aligned}$$

Let $a_1 = Ma_1'$ and $b_1 = \overline{N}b_1'$. Then, $M\overline{N}t \mid [M\overline{N}b_1' x w_2 - NMa_1' w_1]$ implies $b_1' \equiv r a_1' w_1 \overline{w_2} N' (\text{mod } t)$ as $rx \equiv 1 (\text{mod } t)$ and $N = \overline{N}N'$. Now, $\gamma_1 = \frac{a_1}{b_1} = \frac{Ma_1'}{\overline{N}b_1'}$. $(a_1, b_1) = 1 \implies (\overline{N}, a_1') = 1$, $(a_1', b_1') = 1$ and $(M, b_1') = 1$. Rename $a_1' = u$ and $b_1' = r'$. Then $(u, \overline{N}) = 1$. If $(u,t) \neq 1$, then $\exists$ a prime p such that $p \mid t$ and $p \mid u$. Thus $p \mid t$, $p \mid N$ and $p \nmid \overline{N}$, since for any prime p dividing t, p also divides N. So $p \mid t$ and $p \mid N'$. But we have established earlier that $(t, N') = 1$. So $(u, t) = 1$. And, $r' \equiv ruw_1\overline{w_2}N' (\text{mod } t) \implies r'x \equiv uw_1\overline{w_2}N' (\text{mod } t)$. Thus,

$$\gamma_1 = \frac{Mu}{\overline{N}r'} \text{ where } (u, \overline{N}) = 1, (u, t) = 1, (u, r') = 1, (M, r') = 1,$$
$$r' \equiv ruw_1\overline{w_2}N' (\text{mod } t). \quad (4.3)$$

We will now use induction to show that $\forall\ k \geqslant 2$,

$$\begin{aligned} &\gamma_k = Mu\overline{m_2}\cdots\overline{m_{k-1}} + \frac{M\overline{N}t\lambda_k}{\overline{m_1}\cdots\overline{m_k}} \text{ for some } \lambda_k \in \mathbb{Z} \\ &(t, \overline{m_k}) = 1. \end{aligned} \quad (4.4)$$

By Equation (8) in [[6]] we have, $Q_3 = Q_2^{\overline{m_2}} - \zeta_2 X^{c_0} Y^{c_1}$ where $\zeta_2 \in K\backslash\{0\}$, $c_0 \in \mathbb{Z}_{>0}$, $0 \leqslant c_1 < \overline{m_1}$. $(\alpha, \beta^x) \cdot Q_3 = \beta^{x\overline{m_2}\,\overline{m_1}} Q_2^{\overline{m_2}} - \zeta_2 \alpha^{c_0} \beta^{xc_1} X^{c_0} Y^{c_1}$. Since Q_3 is an eigenfunction for H, we have

$$\begin{aligned} \beta^{x\overline{m_2}\,\overline{m_1}} = \alpha^{c_0}\beta^{xc_1} &\implies \beta^{x[\overline{m_2}\,\overline{m_1} - c_1]} = \alpha^{c_0} \\ &\implies \frac{x[\overline{m_2}\,\overline{m_1} - c_1]w_2}{Nt} - \frac{c_0 w_1}{Mt} \in \mathbb{Z} \text{ by Remark 4.5} \\ &\implies M\overline{N}t \,|\, [M\overline{N}r'xw_2\overline{m_2} - Mxw_2c_1 - Nc_0w_1] \text{ as } \overline{m_1} = \overline{N}r' \\ &\implies M \mid c_0 \text{ and } \overline{N} \mid c_1 \text{ as } (M, N) = 1,\ (M, w_1) = 1, \\ &\qquad\qquad (\overline{N}, w_2) = 1,\ (\overline{N}, x) = 1. \end{aligned}$$

Let $c_0 = Mc_0'$ and $c_1 = \overline{N}c_1'$. Plugging them in the above expression and using (4.3), we obtain,

$$\begin{aligned} &M\overline{N}t \mid [M\overline{N}r'xw_2\overline{m_2} - Mxw_2\overline{N}c_1' - NMc_0'w_1] \\ &\implies r'xw_2\overline{m_2} \equiv [w_1c_0'N' + xw_2c_1'] (\text{mod } t) \\ &\implies uw_1\overline{m_2}N' \equiv [w_1c_0'N' + xw_2c_1'] (\text{mod } t) \\ &\implies r'u\overline{m_2} \equiv [r'c_0' + uc_1'] (\text{mod } t). \end{aligned}$$

So, $\overline{m_2}\gamma_2 = c_0 + c_1\gamma_1 = Mc_0' + \overline{N}c_1'\frac{Mu}{\overline{N}r'} = M[\frac{c_0'r' + c_1'u}{r'}] = M[\frac{r'u\overline{m_2} + \lambda_2 t}{r'}] = Mu\overline{m_2} + \frac{M\overline{N}t\lambda_2}{\overline{m_1}}$ for some $\lambda_2 \in \mathbb{Z}$. Thus, $\gamma_2 = Mu + \frac{M\overline{N}t\lambda_2}{\overline{m_1}\,\overline{m_2}}$.
We will now show $(t, \overline{m_2}) = 1$. Suppose if possible $\exists$ a prime p such that $p \mid t$ and $p \mid \overline{m_2}$. Let $\overline{m_2} = pM_2$. So, $\gamma_2 = Mu + \frac{M\overline{N}t\lambda_2}{\overline{m_1}\,\overline{m_2}} \implies \overline{m_2}\gamma_2 = Mu\overline{m_2} + \frac{M\overline{N}t\lambda_2}{\overline{m_1}} \implies pM_2\gamma_2 = pMuM_2 + \frac{Mt\lambda_2}{r'} \implies r'M_2\gamma_2 = r'MuM_2 + M\lambda_2\frac{t}{p}$.
$(w_1, t) = 1$. $(N', t) = 1$. $rx \equiv 1 (\text{mod } t)$ implies $(r, t) = 1$. $w_2\overline{w_2} \equiv 1 (\text{mod } t)$ implies $(\overline{w_2}, t) = 1$. And, $(u, t) = 1$ by (4.3). So, $r' \equiv ruw_1\overline{w_2}N' (\text{mod } t) \implies (r', t) = 1$. So $\exists\ r_1 \in \mathbb{Z}$ such that $r_1r' \equiv 1 (\text{mod } t)$. So in particular, $r_1r' \equiv$

$1(\text{mod } p)$ $\forall$ prime p dividing t. We then have,

$$\begin{aligned} & r_1 r' M_2 \gamma_2 = r_1 r' M u M_2 + r_1 M \lambda_2 \frac{t}{p} \\ & \Longrightarrow (1+\mu_2 p) M_2 \gamma_2 = r_1 r' M u M_2 + r_1 M \lambda_2 \frac{t}{p} \text{ for some } \mu_2 \in \mathbb{Z} \\ & \Longrightarrow M_2 \gamma_2 + \mu_2 \overline{m_2} \gamma_2 \in \mathbb{Z} \subset G(\gamma_0, \gamma_1) \Longrightarrow M_2 \gamma_2 \in G(\gamma_0, \gamma_1). \end{aligned}$$

But this contradicts the minimality of $\overline{m_2}$. So for any prime p dividing t, we have $p \nmid \overline{m_2}$. Thus $(t, \overline{m_2}) = 1$. We now have the induction step for $k = 2$. Suppose (4.4) is true for $k = 3, \cdots, l-1$. By Equation (8) in [[6]] we have, $Q_{l+1} = Q_l^{\overline{m_l}} - \zeta_l X^{c_0} Y^{c_1} Q_2^{c_2} \cdots Q_{l-1}^{c_{l-1}}$ where $\zeta_l \in K \setminus \{0\}$, $c_0 \in \mathbb{Z}_{>0}$, $0 \leqslant c_k < \overline{m_k}$ $\forall$ $k = 1, \cdots, l-1$ and $\overline{m_l}\gamma_l = \sum_{k=0}^{l-1} c_k \gamma_k$. By 2) of Section 2 we have $d(l) = \prod_{k=1}^{l-1} \overline{m_k}$ $\forall$ $l \geqslant 2$. Again, $\overline{m_1} = \overline{N} r'$ by (4.3). So $\forall$ $l \geqslant 2$, $d(l) = \overline{N} r' \overline{d(l)}$, where $\overline{d(l)} = \frac{d(l)}{\overline{m_1}}$. Thus, $\forall$ $l \geqslant 3, \overline{d(l)} = \prod_{k=2}^{l-1} \overline{m_k}$.
Now, $(\alpha, \beta^x) \cdot Q_{l+1} = \beta^{x \overline{m_l} d(l)} Q_l^{\overline{m_l}} - \zeta_l \alpha^{c_0} \beta^{x[\sum_{k=1}^{l-1} c_k d(k)]} X^{c_0} Y^{c_1} Q_2^{c_2} \cdots Q_{l-1}^{c_{l-1}}$. Since Q_{l+1} is an eigenfunction for H we have

$$\begin{aligned} & \Longrightarrow \beta^{x[d(l+1) - \sum_{k=1}^{l-1} c_k d(k)]} = \alpha^{c_0} \\ & \Longrightarrow \frac{x w_2 [d(l+1) - \sum_{k=1}^{l-1} c_k d(k)]}{Nt} - \frac{c_0 w_1}{Mt} \in \mathbb{Z} \text{ by Remark 4.5} \\ & \Longrightarrow M\overline{N}t \mid [M x w_2 \overline{N} r' \overline{d(l+1)} - M x w_2 c_1 - M x w_2 \overline{N} r' \sum_{k=2}^{l-1} c_k \overline{d(k)} - N c_0 w_1] \\ & \Longrightarrow M \mid c_0 \text{ and } \overline{N} \mid c_1 \text{ as } (M,N)=1,\ (M,w_1)=1,\ (\overline{N},x)=1,\ (\overline{N},w_2)=1. \end{aligned}$$

Let $c_0 = M c_0'$ and $c_1 = \overline{N} c_1'$. Plugging them in the above expression, and using (4.3), we obtain

$$\begin{aligned} & M\overline{N}t \mid [M x w_2 \overline{N} r' \overline{d(l+1)} - M x w_2 \overline{N} c_1' - M x w_2 \overline{N} r' \sum_{k=2}^{l-1} c_k \overline{d(k)} - N M w_1 c_0'] \\ & \Longrightarrow t \mid [x w_2 r' \overline{d(l+1)} - x w_2 c_1' - x w_2 r' \sum_{k=2}^{l-1} c_k \overline{d(k)} - w_1 c_0' N'] \\ & \Longrightarrow r' x w_2 \overline{d(l+1)} \equiv [c_0' w_1 N' + c_1' x w_2 + r' x w_2 \sum_{k=2}^{l-1} c_k \overline{d(k)}] (\text{mod } t) \\ & \Longrightarrow r' u \overline{d(l+1)} \equiv [r' c_0' + c_1' u + r' u \sum_{k=2}^{l-1} c_k \overline{d(k)}] (\text{mod } t). \end{aligned}$$

Now,

$$
\begin{aligned}
\overline{m_l}\gamma_l &= c_0 + c_1\gamma_1 + \sum_{k=2}^{l-1} c_k\gamma_k \\
&= Mc_0' + \overline{N}c_1'\frac{Mu}{\overline{N}r'} + \sum_{k=2}^{l-1} c_k[Mu\overline{d(k)} + \frac{M\overline{N}t\lambda_k}{d(k+1)}] \\
&\qquad \text{where } \lambda_k \in \mathbb{Z}, \text{ by induction hypothesis} \\
&= M[\frac{c_0'r' + c_1'u + r'u\sum_{k=2}^{l-1} c_k\overline{d(k)}}{r'} + \frac{\overline{N}t\theta_l}{d(l)}] \\
&\qquad \text{for some } \theta_l \in \mathbb{Z}, \text{ as } i_1 \leqslant i_2 \Longrightarrow d(i_1) \mid d(i_2) \\
&= M[\frac{r'u\overline{d(l+1)} + \mu_l t}{r'} + \frac{\overline{N}t\theta_l}{d(l)}] \text{ for some } \mu_l \in \mathbb{Z} \\
&= Mu\overline{d(l+1)} + \frac{M\overline{N}t\mu_l}{\overline{m_1}} + \frac{M\overline{N}t\theta_l}{d(l)} = Mu\overline{d(l+1)} + \frac{M\overline{N}t\lambda_l}{d(l)} \\
&\qquad \text{for some } \lambda_l \in \mathbb{Z} \\
&\Longrightarrow \gamma_l = Mu\overline{m_2}\cdots\overline{m_{l-1}} + \frac{M\overline{N}t\lambda_l}{\overline{m_1}\cdots\overline{m_l}}.
\end{aligned}
$$

By our induction hypothesis, $(t, \overline{m_k}) = 1 \ \forall \ k = 2, \cdots, l-1$. So $(p, \overline{m_k}) = 1$ for any prime p dividing t, $\forall \ k = 2, \cdots, l-1$, hence, $(p, \overline{d(l)}) = 1$. Suppose if possible $\exists$ a prime $p \mid t$ such that $p \mid \overline{m_l}$. Let $\overline{m_l} = pM_l$. Now, $(r', t) = 1 \Longrightarrow (r', p) = 1$. So $(p, r'\overline{d(l)}) = 1$. So $\exists \ r_l \in \mathbb{Z}$ such that $r_l r'\overline{d(l)} \equiv 1 (\text{mod } p)$. Let $r_l r'\overline{d(l)} = 1 + \mu_l p$ for some $\mu_l \in \mathbb{Z}$. Now,

$$
\begin{aligned}
\gamma_l &= Mu\overline{m_2}\cdots\overline{m_{l-1}} + \frac{M\overline{N}t\lambda_l}{\overline{m_1}\cdots\overline{m_l}} \\
&\Longrightarrow pM_l\gamma_l = Mu\overline{m_2}\cdots\overline{m_l} + \frac{Mt\lambda_l}{r'\overline{d(l)}} \text{ as } \overline{m_l} = pM_l,\ \overline{m_1} = \overline{N}r',\ \overline{d(l)} = \prod_{k=2}^{l-1}\overline{m_k} \\
&\Longrightarrow r'\overline{d(l)}M_l\gamma_l = r'\overline{d(l)}Mu\overline{m_2}\cdots\overline{m_{l-1}}M_l + M\lambda_l\frac{t}{p} \text{ as } \overline{m_l} = pM_l \\
&\Longrightarrow r_l r'\overline{d(l)}M_l\gamma_l = r_l r'\overline{d(l)}Mu\overline{m_2}\cdots\overline{m_{l-1}}M_l + r_l M\lambda_l\frac{t}{p} \in \mathbb{Z} \\
&\Longrightarrow (1 + \mu_l p)M_l\gamma_l \in \mathbb{Z} \Longrightarrow M_l\gamma_l + \mu_l\overline{m_l}\gamma_l \in \mathbb{Z} \subset G(\gamma_0, \cdots, \gamma_{l-1}) \\
&\Longrightarrow M_l\gamma_l \in G(\gamma_0, \cdots, \gamma_{l-1}).
\end{aligned}
$$

But this contradicts the minimality of $\overline{m_l}$. So for any prime p dividing t, we have $p \nmid \overline{m_l}$. Thus $(t, \overline{m_l}) = 1$. We now have the induction step for $k = l$.

$(t,r')=1 \implies \overline{N}t \nmid \overline{N}r' \implies Nt \nmid \overline{N}r' \implies n \nmid \overline{m_1} \implies n \nmid d(2)$. From the induction we have $(t,\overline{m_k})=1\ \forall\ k\geqslant 2$. Thus $(t,\prod_{k=2}^{l-1}\overline{m_k})=1 \implies (t,\overline{d(l)})=1\ \forall\ l\geqslant 3 \implies (t,r'\overline{d(l)})=1\ \forall\ l\geqslant 3$. $t\nmid r'\overline{d(l)}\ \forall\ l\geqslant 3 \implies \overline{N}t \nmid \overline{N}r'\overline{d(l)}\ \forall\ l\geqslant 3 \implies Nt \nmid \overline{m_1}\overline{d(l)}\ \forall\ l\geqslant 3 \implies n\nmid d(l)\ \forall\ l\geqslant 3$. So together we have, $n\nmid d(l)\ \forall\ l\geqslant 2$. Thus by Lemma 4.3, we have $S^{R_{\mathfrak{m}}}(\nu)$ is not finitely generated over $S^{A_{\mathfrak{n}}}(\nu)$. □

We are now ready to prove Theorem 4.1.

Proof. — Let $H\leqslant \mathbb{U}_m\times\mathbb{U}_n$ be as in Remark 1.3 and suppose that ν is a rational rank 1 non discrete valuation dominating $R_{\mathfrak{m}}$ with a generating sequence (2.2) of eigenfunctions for H. By Proposition 4.4, we have $t\geqslant (m,n)$. Since $t\mid m$ and $t\mid n$, we have $(m,n)=t$.

Conversely, let H be as in Remark 1.3 and suppose that $(m,n)=t$. We will show that $\exists$ a rational rank 1 non discrete valuation dominating $R_{\mathfrak{m}}$ with a generating sequence (2.2) of eigenfunctions for H. We consider the cases $t=1$ and $t>1$ separately.

Suppose that $(m,n)=t=1$. We will construct a rational rank 1 non discrete valuation ν dominating $R_{\mathfrak{m}}$, with a generating sequence (2.2) of eigenfunctions for H. Let $\{q_l\}_{l\geqslant 2}$ be an infinite family of distinct prime numbers, such that $(q_l,m)=1$, $(q_l,n)=1$ for all $l\geqslant 2$. Let $q_1=n$. Let $\{c_l\}_{l\geqslant 1}\in\mathbb{Z}_{>0}$ be positive integers such that

$$c_1=m,\ c_l\equiv 0(\text{mod } m)\ \ \forall\ l\geqslant 1$$
$$c_{l+1}>q_{l+1}c_l\ \ \forall\ l\geqslant 1,\ (c_l,q_l)=1\ \ \forall\ l\geqslant 1.$$

We define a sequence of positive rational numbers $\{\gamma_l\}_{l\geqslant 0}$ as $\gamma_0=1$, $\gamma_l=\frac{c_l}{q_l}\ \ \forall\ l\geqslant 1$. We will show $\overline{m_l}=q_l\ \ \forall\ l\geqslant 1$, where $\overline{m_l}=\min\{q\in\mathbb{Z}_{>0}\,|\,q\gamma_l\in G(\gamma_0,\cdots,\gamma_{l-1})\}$. Now, $\gamma_1=\frac{c_1}{q_1}=\frac{m}{n}$. Since $(m,n)=1$, we have $\overline{m_1}=n=q_1$. For $l\geqslant 2$, $q_l\gamma_l=c_l\in\mathbb{Z}\implies 1\leqslant\overline{m_l}\leqslant q_l$. Suppose $q\in\mathbb{Z}_{>0}$ such that $q\gamma_l=q\frac{c_l}{q_l}=\sum_{k=0}^{l-1}a_k\gamma_k=\sum_{k=0}^{l-1}a_k\frac{c_k}{q_k}$. Then $q_l\mid qc_l\prod_{k=1}^{l-1}q_k$, that is, $q_l\mid qc_ln\prod_{k=2}^{l-1}q_k$. Now, $(q_l,c_l)=1$ and $(q_l,n)=1$. Again, $(q_l,q_k)=1\ \ \forall\ k\neq l$, as they are distinct primes. So, $q_l\,|\,q$. Thus we have $\overline{m_l}=q_l\ \ \forall\ l\geqslant 1$. And, $\overline{m_l}\gamma_l=q_l\gamma_l=c_l<\frac{c_{l+1}}{q_{l+1}}=\gamma_{l+1}$. Thus we have a sequence of positive rational numbers $\{\gamma_l\}_{l\geqslant 0}$, such that $\gamma_{l+1}>\overline{m_l}\gamma_l\ \ \forall\ l\geqslant 1$. By Theorem 1.2 of [[6]], since $R_{\mathfrak{m}}$ is a regular local ring of dimension 2, there is a valuation ν dominating $R_{\mathfrak{m}}$, such that $S^{R_{\mathfrak{m}}}(\nu)=S(\gamma_0,\gamma_1,\cdots)$. ν is a rational rank 1 non discrete valuation by the construction. By Theorem 4.2 of [[6]], $\exists$ a generating sequence (2.2) $\{Q_l\}_{l\geqslant 0}, Q_0=X, Q_1=Y,\cdots$ such that $\nu(Q_l)=\gamma_l\ \ \forall\ l\geqslant 0$.
From the recursive construction of the $\{\gamma_l\}_{l\geqslant 0}$, we have the generating sequence as $Q_0=X$, $Q_1=Y$, $Q_2=Y^n-\lambda_1X^m$, where $\lambda_1\in K\backslash\{0\}$. For all $l\geqslant$

2, $Q_{l+1} = Q_l^{q_l} - \lambda_l X^{f_0} Y^{f_1} \cdots Q_{l-1}^{f_{l-1}}$, where $q_l\gamma_l = c_l = f_0 + \sum_{k=1}^{l-1} f_k\gamma_k$, $0 \leqslant f_k < \overline{m_k}$ $\forall$ $k \geqslant 1$. Now, $(c_k, q_k) = 1$ $\forall$ $k \geqslant 1$, and $(q_k, q_h) = 1$ $\forall$ $k \neq h$. So, $c_l = f_0 + \sum_{k=1}^{l-1} \frac{f_k c_k}{q_k} \Longrightarrow c_l \prod_{k=1}^{l-1} q_k = f_0 \prod_{k=1}^{l-1} q_k + \frac{f_1 c_1 \prod_{k=1}^{l-1} q_k}{q_1} + \cdots + \frac{f_{l-1} c_{l-1} \prod_{k=1}^{l-1} q_k}{q_{l-1}}$, which implies $q_k \mid f_k$ $\forall$ $k \geqslant 1$. Since $0 \leqslant f_k < \overline{m_k} = q_k$, this implies $f_k = 0$ $\forall$ $k \geqslant 1$. So we have the generating sequence as,

$$Q_0 = X,\ Q_1 = Y,\ Q_2 = Y^n - \lambda_1 X^m,\ Q_{l+1} = Q_l^{q_l} - \lambda_l X^{c_l} \quad \forall\ l \geqslant 2$$
$$\text{where } \lambda_l \in K \setminus \{0\}\ \forall\ l \geqslant 1.$$

We now show that each Q_l is an eigenfunction for $H = \{(\alpha^a, \beta^b) \mid a, b \in \mathbb{Z}\}$. For all $l \geqslant 2$, $d(l) = \prod_{k=1}^{l-1} \overline{m_k} = q_1 \cdots q_{l-1} = nq_2 \cdots q_{l-1}$. We have, $(\alpha^a, \beta^b) \cdot Q_2 = \beta^{bn} Y^n - \lambda_1 \alpha^{am} X^m = Q_2$. So, Q_2 is an eigenfunction. Suppose $Q_3, \cdots, Q_l$ are eigenfunctions for H. We check for Q_{l+1}. From (3.2), $(\alpha^a, \beta^b) \cdot Q_k = \beta^{bd(k)} Q_k$ $\forall$ $2 \leqslant k \leqslant l$. Since $m \mid c_l$ and $n \mid d(l)$, we have $(\alpha^a, \beta^b) \cdot Q_{l+1} = \beta^{bq_l d(l)} Q_l^{q_l} - \lambda_l \alpha^{ac_l} X^{c_l} = Q_{l+1}$. Thus Q_{l+1} is an eigenfunction. Thus by induction, $\{Q_l\}_{l \geqslant 0}$ is a generating sequence of eigenfunctions for H.

Now we consider the case $(m, n) = t > 1$. We will construct a rational rank 1 non discrete valuation ν dominating $R_{\mathfrak{m}}$, with a generating sequence (2.2) of eigenfunctions for H.

Since $(t, x) = 1$, there are positive integers r, s such that $rx - st = 1$. So $(r, t) = 1$. From Lemma 3 in §2, Chapter III of [[12]], we have that if r, t are positive integers such that $(r, t) = 1$, then there are infinitely many prime numbers of the form $r + \theta t$, where $\theta \in \mathbb{N}$. Define the family $\mathfrak{R} = \{r^{(k)}\}_{k \geqslant 0}$ as $r^{(0)} = r$, $r^{(k)} = k$-th prime in the above prime series. Any two elements in the family $\mathfrak{R}$ are coprime by construction. Also, $r^{(k)} = r + \theta_k t \Longrightarrow r^{(k)} \equiv r(\text{mod } t)$ $\forall$ k. Since $\mathfrak{R}$ is an infinite family such that any two elements in $\mathfrak{R}$ are mutually prime, it follows that there is an infinite ordered family of distinct prime numbers $\mathfrak{F} = \{r_l\}_{l \geqslant 1}$ such that, $r_l \equiv r(\text{mod } t)$, $(r_l, \frac{m}{t}) = 1$, $(r_l, \frac{n}{t}) = 1$, $(r_l, w_1) = 1$, $(r_l, w_2) = 1$ $\forall$ $l \geqslant 1$, where w_1 and w_2 are as in Remark 4.5. Let $d = (w_1, w_2)$. Thus $(\frac{w_1}{d}, \frac{w_2}{d}) = 1$. Define two sequences $(a_l)_{l \geqslant 1}$ and $(b_l)_{l \geqslant 1}$ of non negative integers as follows,

$$b_1 = 0,\ r_l \mid b_l \quad \forall\ l \geqslant 2,\ t \mid b_l \quad \forall\ l \geqslant 2$$
$$b_{l+1} > r_{l+1}[r^{l-1} + b_l] - r^l \quad \forall\ l \geqslant 1$$
$$a_l = \frac{m}{t}[r^{l-1} + b_l]\frac{w_2}{d} \quad \forall\ l \geqslant 1.$$

Here $r_l \in \mathfrak{F}\ \forall\ l \geqslant 1$. Define a sequence of positive rational numbers $\{\gamma_l\}_{l\geqslant 0}$ as follows

$$\gamma_0 = 1,\ \gamma_1 = \frac{\frac{m}{t}\frac{w_2}{d}}{r_1\frac{n}{t}\frac{w_1}{d}},$$

$$\gamma_l = \frac{a_l}{r_l} = \frac{m}{t}[\frac{r^{l-1}+b_l}{r_l}]\frac{w_2}{d}\quad \forall\ l \geqslant 2.$$

We will show $\overline{m_1} = r_1\frac{n}{t}\frac{w_1}{d}$ and $\overline{m_l} = r_l\ \ \forall\ l \geqslant 2$, where $\overline{m_l} = \min\{q \in \mathbb{Z}_{>0} \,|\, q\gamma_l \in G(\gamma_0, \cdots, \gamma_{l-1})\}$. $(\frac{w_1}{d}, \frac{w_2}{d}) = 1$, $(r_1, \frac{w_2}{d}) = 1$ and $(\frac{n}{t}, \frac{w_2}{d}) = 1$ implies $(\frac{w_2}{d}, r_1\frac{n}{t}\frac{w_1}{d}) = 1$. Also, $(\frac{m}{t}, \frac{n}{t}) = 1$, $(\frac{m}{t}, r_1) = 1$ and $(\frac{m}{t}, \frac{w_1}{d}) = 1$ implies $(\frac{m}{t}, r_1\frac{n}{t}\frac{w_1}{d}) = 1$. Thus, $(\frac{w_2}{d}\frac{m}{t}, r_1\frac{n}{t}\frac{w_1}{d}) = 1$, hence $\overline{m_1} = r_1\frac{n}{t}\frac{w_1}{d}$.
Now $\forall\ l \geqslant 2$, $r_l\gamma_l = a_l \in \mathbb{Z} \implies 1 \leqslant \overline{m_l} \leqslant r_l$. Suppose $\exists$ a positive integer q such that $q\gamma_l \in G(\gamma_0, \cdots, \gamma_{l-1})$. Then $q\gamma_l = q\frac{a_l}{r_l} = c_0 + c_1\frac{a_1}{r_1\frac{n}{t}\frac{w_1}{d}} + \sum_{k=2}^{l-1} c_k\frac{a_k}{r_k}$, where $c_k \in \mathbb{Z}\ \forall\ k = 0, \cdots, l-1$. Thus $r_l \mid qa_l\frac{n}{t}\frac{w_1}{d}\prod_{k=1}^{l-1} r_k$. Now, $(r_l, \frac{n}{t}) = 1$, and $(r_l, r_k) = 1\ \forall\ k \neq l$, as they are distinct primes. Also, $(r_l, \frac{w_1}{d}) = 1$. So, $r_l \mid qa_l$. And, $r_l > r \implies r_l \nmid r \implies r_l \nmid \frac{m}{t}[r^{l-1}+b_l]\frac{w_2}{d} = a_l$ as $(r_l, \frac{w_2}{d}) = 1$, $(r_l, \frac{m}{t}) = 1$ and $r_l \mid b_l$. Thus, $r_l \mid q$. Hence we have $\overline{m_1} = r_1\frac{n}{t}\frac{w_1}{d}$ and $\overline{m_l} = r_l\ \ \forall\ l \geqslant 2$.
Now, $b_{l+1} > r_{l+1}[r^{l-1}+b_l] - r^l\ \forall\ l \geqslant 1$ and $b_1 = 0$ implies $b_2 > r_2 - r$. Thus, $a_2 = \frac{m}{t}[r + b_2]\frac{w_2}{d} > r_2\frac{m}{t}\frac{w_2}{d} \implies \gamma_2 = \frac{a_2}{r_2} > \frac{m}{t}\frac{w_2}{d} = \overline{m_1}\gamma_1$. For $l \geqslant 2$, we have $r^l + b_{l+1} > r_{l+1}[r^{l-1} + b_l] \implies \frac{m}{t}[r^l + b_{l+1}]\frac{w_2}{d} > r_{l+1}\frac{m}{t}[r^{l-1} + b_l]\frac{w_2}{d} \implies \gamma_{l+1} = \frac{a_{l+1}}{r_{l+1}} > a_l = \overline{m_l}\gamma_l$.
Thus we have a sequence of positive rational numbers $\{\gamma_l\}_{l\geqslant 0}$ such that $\gamma_{l+1} > \overline{m_l}\gamma_l\ \ \forall\ l \geqslant 1$. By Theorem 1.2 of [[6]], since $R_\mathfrak{m}$ is a regular local ring of dimension 2, there is a valuation ν dominating $R_\mathfrak{m}$, such that $S^{R_\mathfrak{m}}(\nu) = S(\gamma_0, \gamma_1, \cdots)$. ν is a rational rank 1 non discrete valuation by the construction. By Theorem 4.2 of [[6]], $\exists$ a generating sequence (2.2) $\{Q_l\}_{l\geqslant 0}, Q_0 = X, Q_1 = Y, \cdots$ such that $\nu(Q_l) = \gamma_l\ \ \forall\ l \geqslant 0$.
From the recursive construction of the $\{\gamma_l\}_{l\geqslant 0}$, we have the generating sequence as $Q_0 = X$, $Q_1 = Y$, $Q_2 = Y^{r_1\frac{n}{t}\frac{w_1}{d}} - \lambda_1 X^{\frac{m}{t}\frac{w_2}{d}}$. For all $l \geqslant 2$, $Q_{l+1} = Q_l^{r_l} - \lambda_l X^{f_0}Y^{f_1}\cdots Q_{l-1}^{f_{l-1}}$, where $0 \leqslant f_k < \overline{m_k}\ \ \forall\ \ k \geqslant 1$ and $r_l\gamma_l = a_l = f_0 + \sum_{k=1}^{l-1} f_k\gamma_k$. So, $a_l = f_0 + \sum_{k=1}^{l-1}\frac{f_k a_k}{\overline{m_k}}$. We observe, from our construction, $(\overline{m_k}, \overline{m_h}) = 1\ \forall\ k \neq h$. Also, $(\overline{m_k}, a_k) = 1\ \forall\ k \geqslant 1$.
Thus, $a_l\prod_{k=1}^{l-1}\overline{m_k} = f_0\prod_{k=1}^{l-1}\overline{m_k} + \frac{f_1 a_1\prod_{k=1}^{l-1}\overline{m_k}}{\overline{m_1}} + \cdots + \frac{f_{l-1}a_{l-1}\prod_{k=1}^{l-1}\overline{m_k}}{\overline{m_{l-1}}} \implies \overline{m_k} \mid f_k\ \ \forall\ \ k \geqslant 1$. Since $0 \leqslant f_k < \overline{m_k}$, we have $f_k = 0\ \forall\ k \geqslant 1$. Thus the generating sequence is given as,

$$Q_0 = X,\ Q_1 = Y,\ Q_2 = Y^{r_1\frac{n}{t}\frac{w_1}{d}} - \lambda_1 X^{\frac{m}{t}\frac{w_2}{d}}$$
$$Q_{l+1} = Q_l^{r_l} - \lambda_l X^{a_l}\ \ \forall\ l \geqslant 2$$
$$\text{where } \lambda_l \in K \setminus \{0\}\ \forall\ l \geqslant 1.$$

This is a minimal generating sequence as $\overline{m_l} > 1 \ \forall \ l \geqslant 1$. We now show that each Q_l is an eigenfunction for H. From (2.1), $(\alpha^a, \beta^b) \cdot Q_2 = \beta^{\frac{r_1 bn}{t}\frac{w_1}{d}} Y^{r_1 \frac{n}{t}\frac{w_1}{d}} - \lambda_1 \alpha^{\frac{am}{t}\frac{w_2}{d}} X^{\frac{m}{t}\frac{w_2}{d}}$. Now, $\forall \ b \equiv ax (\text{mod } t)$, $r_1 b \equiv a (\text{mod } t)$, hence, $(\frac{r_1 b - a}{t})(\frac{w_1 w_2}{d}) \in \mathbb{Z}$. Thus by Remark 4.5, $\beta^{\frac{r_1 bn}{t}\frac{w_1}{d}} = \alpha^{\frac{am}{t}\frac{w_2}{d}} \ \forall \ b \equiv ax(\text{mod } t)$, that is, Q_2 is an eigenfunction for H.
Suppose $Q_3, \cdots, Q_l$ are eigenfunctions for $H_{i,j,t,x}$. We check for Q_{l+1}. We note $d(k) = \overline{m_1} \cdots \overline{m_{k-1}} = \frac{n}{t}\frac{w_1}{d} r_1 r_2 \cdots r_{k-1}$. From (3.2) we have, $(\alpha^a, \beta^b) \cdot Q_k = \beta^{bd(k)} Q_k \ \forall \ 1 \leqslant k \leqslant l$. Now, $(\alpha^a, \beta^b) \cdot Q_{l+1} = \beta^{\frac{bnr_1 \cdots r_l}{t}\frac{w_1}{d}} Q_l^{r_l} - \lambda_l \alpha^{aa_l} X^{a_l}$. Since $r_k \equiv r (\text{mod } t) \ \forall \ k \geqslant 1$, $rx \equiv 1 (\text{mod } t)$ and $t \mid b_l$, we have

$$
\begin{aligned}
&\frac{br_1 \cdots r_l}{t} - \frac{ar^{l-1}}{t} \in \mathbb{Z} \ \forall \ b \equiv ax(\text{mod } t) \\
\Longrightarrow &\frac{br_1 \cdots r_l}{t} - \frac{a[r^{l-1} + b_l]}{t} \in \mathbb{Z} \ \forall \ b \equiv ax(\text{mod } t) \\
\Longrightarrow &\frac{br_1 \cdots r_l}{t}(\frac{w_1 w_2}{d}) - \frac{a[r^{l-1} + b_l]}{t}(\frac{w_1 w_2}{d}) \in \mathbb{Z} \ \forall \ b \equiv ax(\text{mod } t) \\
\Longrightarrow &\frac{bnr_1 \cdots r_l}{t}(\frac{w_1 w_2}{dn}) - \frac{am[r^{l-1} + b_l]}{t}(\frac{w_1 w_2}{dm}) \in \mathbb{Z} \ \forall \ b \equiv ax(\text{mod } t) \\
\Longrightarrow &(\frac{bnr_1 \cdots r_l}{t}\frac{w_1}{d})\frac{w_2}{n} - (aa_l)\frac{w_1}{m} \in \mathbb{Z} \ \forall \ b \equiv ax(\text{mod } t).
\end{aligned}
$$

Thus, by Remark 4.5, $\beta^{\frac{bnr_1 \cdots r_l}{t}\frac{w_1}{d}} = \alpha^{aa_l}$ for all $b \equiv ax(\text{mod } t)$, and hence Q_{l+1} is an eigenfunction for H. Thus by induction, $\{Q_l\}_{l \geqslant 0}$ is a minimal generating sequence of eigenfunctions for H. This completes the proof of part 1) of Theorem 4.1.

Now we suppose $(m, n) = t = 1$ and ν is a rational rank 1 non discrete valuation dominating $R_{\mathfrak{m}}$ with a generating sequence (2.2) of eigenfunctions for H. Let $\nu(Q_l) = \gamma_l \ \forall \ l \in \mathbb{N}$. We have $Q_0 = X, Q_1 = Y$. By Equation (8) in [[6]], $Q_2 = Y^s - \lambda X^r$ where $\lambda \in K \setminus \{0\}$, $s\gamma_1 = r\gamma_0$. Since $(m, n) = 1$, by Chinese Remainder Theorem (Theorem 2.1, §2, [[9]]) we have H is a cyclic group, generated by (α, β). By (2.1) we have $(\alpha, \beta) \cdot Q_2 = \beta^s Y^s - \lambda \alpha^r X^r$. Since Q_2 is an eigenfunction, we have

$$
\begin{aligned}
\beta^s = \alpha^r &\Longrightarrow \frac{sw_2}{n} - \frac{rw_1}{m} \in \mathbb{Z} \text{ by Remark 4.5} \\
&\Longrightarrow m \mid r \text{ and } n \mid s \text{ as } (m, w_1) = 1,\ (n, w_2) = 1,\ (m, n) = 1.
\end{aligned}
$$

So, $Q_2 = Y^s - \lambda X^r \in K[X^m, Y^n] \subset A$. Thus by Proposition 4.2, we have part 2) of Theorem 4.1.

We observe that the part 3) of Theorem 4.1 follows from Propositions 4.6 and 4.7. This completes the proof of Theorem 4.1. □

Example 4.8. — Let $m > 1$. Let $(c_1, m) = 1$ and $(c_2, m) = 1$. Let $\mathbb{U}_m$ acts on $R = K[X, Y]$ by the diagonal action given by K-algebra isomorphisms satisfying $\alpha \cdot X^r Y^s = \alpha^{c_1 r + c_2 s} X^r Y^s$. Suppose ν is a rational rank 1 nondiscrete valuation dominating $R_{\mathfrak{m}}$. Let $\{Q_l\}_{l \geqslant 0}$ be the generating sequence (2.2) of the valuation ν, where $Q_0 = X, Q_1 = Y$, and suppose that each Q_l is an eigenfunction for $\mathbb{U}_m$ under the diagonal action. Let $B = R^{\mathbb{U}_m}$ and $\mathfrak{b} = B \cap \mathfrak{m}$. Then $S^{R_{\mathfrak{m}}}(\nu)$ is not finitely generated over $S^{B_{\mathfrak{b}}}(\nu)$.

Proof. — α is a primitive m-th root of unity, and $(c_1, m) = (c_2, m) = 1$. So $\mathbb{U}_m = < \alpha > = < \alpha^{c_1} > = < \alpha^{c_2} >$. Now, the subdirect product $H \leqslant \mathbb{U}_m \times \mathbb{U}_m$ is given by

$$H = \{((\alpha^{c_1})^a, (\alpha^{c_2})^b) \mid b \equiv a (\text{mod } m)\} =< (\alpha^{c_1}, \alpha^{c_2}) > .$$

From (2.1), we have H acts on R by K-algebra isomorphisms satisfying $(\alpha^{c_1}, \alpha^{c_2}) \cdot X^r Y^s = \alpha^{c_1 r + c_2 s} X^r Y^s$. Thus we have, $\alpha \cdot X^r Y^s = (\alpha^{c_1}, \alpha^{c_2}) \cdot X^r Y^s$.
Now let $\{Q_l\}_{l \geqslant 0}$ be the generating sequence (2.2) of the valuation ν, where $Q_0 = X, Q_1 = Y$, and each Q_l is an eigenfunction for $\mathbb{U}_m$ under the diagonal action. Hence each Q_l is thus an eigenfunction for H. And, $B = R^{\mathbb{U}_m} = R^H = A$. Also $\mathfrak{b} = B \cap \mathfrak{m} = A \cap \mathfrak{m} = \mathfrak{n}$.
Using the same notation as in Theorem 4.1, we have $t = m$. Since $m > 1$, by Theorem 4.1 we have $S^{R_{\mathfrak{m}}}(\nu)$ is not finitely generated over $S^{A_{\mathfrak{n}}}(\nu)$. Hence, $S^{R_{\mathfrak{m}}}(\nu)$ is not finitely generated over $S^{B_{\mathfrak{b}}}(\nu)$.
When $m = 2$, $c_1 = c_2 = 1$, this is Example 9.3 of [[6]]. □

5. Non-splitting

Suppose that a local domain B dominates a local domain A. Let L be the quotient field of A and M be the quotient field of B. Suppose ω is a valuation of L which dominates A. We say that ω does not split in B if there is a unique extension ω^* of ω to M which dominates B.

We use the same notation as in the previous sections.

Theorem 5.1. — *Let $H \leqslant \mathbb{U}_m \times \mathbb{U}_n$ be as in Remark 1.3 such that $(m, n) = t$. Let assumptions be as in Theorem 3.1. Let $\overline{\nu} = \nu \mid_{Q(A)}$ where $Q(A)$ denotes the quotient field of A. Then $\overline{\nu}$ does not split in $R_{\mathfrak{m}}$.*

Proof. — Let $\{Q_k\}_{k \geqslant 0}$, $\{\gamma_k\}_{k \geqslant 0}$ and $\{\overline{m_k}\}_{k \geqslant 1}$ be as in Section 2. Thus $Q_0 = X$ and $Q_1 = Y$. Without any loss of generality, we can assume $\gamma_0 = 1$. Set $m = Mt$ and $n = Nt$ where $M, N \in \mathbb{Z}_{>0}$ and $(M, N) = 1$. From (3.1) we have

$$S^{A_\mathfrak{n}}(\nu)=\left\{ l\gamma_0+j_1\gamma_1+\cdots+j_r\gamma_r \;\middle|\; \begin{array}{l} l\in\mathbb{N},\ r\in\mathbb{N},\ 0\leqslant j_k<\overline{m_k}\ \forall\ k=1,\cdots,r \\ \alpha^{la}\beta^{b\sum_{k=1}^{r}[j_k d(k)]}=1 \\ \forall\ b\equiv ax(\text{mod } t) \end{array} \right\}.$$

Now, $\overline{\nu}=\nu\mid_{Q(A)}$. Thus $S^{A_\mathfrak{n}}(\nu)=\{\nu(f)\mid 0\neq f\in A_\mathfrak{n}\}=S^{A_\mathfrak{n}}(\overline{\nu})$. The group generated by $S^{A_\mathfrak{n}}(\overline{\nu})$ is $\Gamma_{\overline{\nu}}$, the value group of $\overline{\nu}$ (1.2, [[3]]). Thus $\Gamma_{\overline{\nu}}=\{s_1-s_2\mid s_1,s_2\in S^{A_\mathfrak{n}}(\nu)\}$.

Suppose $\gamma_0\in\Gamma_{\overline{\nu}}$. Then we have a representation,

$$\gamma_0=(l_1\gamma_0+\sum_{k=1}^{r}h_{1,k}\gamma_k)-(l_2\gamma_0+\sum_{k=1}^{r}h_{2,k}\gamma_k)=(l_1-l_2)\gamma_0+\sum_{k=1}^{r}(h_{1,k}-h_{2,k})\gamma_k$$

where $l_1\gamma_0+\sum_{k=1}^{r}h_{1,k}\gamma_k\in S^{A_\mathfrak{n}}(\nu)$, and $l_2\gamma_0+\sum_{k=1}^{r}h_{2,k}\gamma_k\in S^{A_\mathfrak{n}}(\nu)$. Thus $l_1,l_2\in\mathbb{N}$, $r\in\mathbb{N}$ and $0\leqslant h_{1,k},h_{2,k}<\overline{m_k}\ \forall\ k=1,\cdots,r$. So, $|h_{1,k}-h_{2,k}|<\overline{m_k}\ \forall\ k=1,\cdots,r$. Now $(h_{1,r}-h_{2,r})\gamma_r\in G(\gamma_0,\cdots,\gamma_{r-1})$ and $|h_{1,r}-h_{2,r}|<\overline{m_r}\implies h_{1,r}=h_{2,r}$. With the same argument, we have $h_{1,k}=h_{2,k}\ \forall\ k=1,\cdots,r$. So in the representation of γ_0, we have $\gamma_0=(l_1-l_2)\gamma_0\implies l_1-l_2=1$. Also,

$$\alpha^{l_1a}\beta^{b\sum_{k=1}^{r}[h_{1,k}d(k)]}=1=\alpha^{l_2a}\beta^{b\sum_{k=1}^{r}[h_{2,k}d(k)]}$$
$$\implies\alpha^{(l_1-l_2)a}\beta^{b\sum_{k=1}^{r}[(h_{1,k}-h_{2,k})d(k)]}=1\ \forall\ b\equiv ax(\text{mod } t).$$

Since $l_1-l_2=1$ and $h_{1,k}=h_{2,k}\ \forall\ k=1,\cdots,r$, we have $\alpha^a=1\ \forall\ b\equiv ax(\text{mod } t)$. Thus $\alpha=1$, that is, $m=1$. So we have obtained,

$$\gamma_0\in\Gamma_{\overline{\nu}}\implies M=1,\ t=1. \tag{5.1}$$

Suppose $\gamma_1\in\Gamma_{\overline{\nu}}$. Then we have a representation,

$$\gamma_1=(l_1\gamma_0+\sum_{k=1}^{r}j_{1,k}\gamma_k)-(l_2\gamma_0+\sum_{k=1}^{r}j_{2,k}\gamma_k)=(l_1-l_2)\gamma_0+\sum_{k=1}^{r}(j_{1,k}-j_{2,k})\gamma_k$$

where $l_1\gamma_0+\sum_{k=1}^{r}j_{1,k}\gamma_k\in S^{A_\mathfrak{n}}(\nu)$, and $l_2\gamma_0+\sum_{k=1}^{r}j_{2,k}\gamma_k\in S^{A_\mathfrak{n}}(\nu)$. So, $l_1,l_2\in\mathbb{N}$, $r\in\mathbb{N}$ and $0\leqslant j_{1,k},j_{2,k}<\overline{m_k}\ \forall\ k=1,\cdots,r$. So, $|j_{1,k}-j_{2,k}|<\overline{m_k}\ \forall\ k=1,\cdots,r$. Now, $(j_{1,r}-j_{2,r})\gamma_r\in G(\gamma_0,\cdots,\gamma_{r-1})$ and $|j_{1,r}-j_{2,r}|<\overline{m_r}\implies j_{1,r}=j_{2,r}$. With the same argument, we have $j_{1,k}=j_{2,k}\ \forall\ k=2,\cdots r$. Thus we have, $\gamma_1=(l_1-l_2)\gamma_0+(j_{1,1}-j_{2,1})\gamma_1$ where $0\leqslant|j_{1,1}-j_{2,1}|<\overline{m_1}$. Again, $\forall\ b\equiv ax(\text{mod } t)$ we have

$$\alpha^{l_1a}\beta^{b\sum_{k=1}^{r}[j_{1,k}d(k)]}=1=\alpha^{l_2a}\beta^{b\sum_{k=1}^{r}[j_{2,k}d(k)]}.$$

Since $d(1)=\deg_Y(Y)=1$ and $j_{1,k}=j_{2,k}\ \forall\ k=2,\cdots,r$, we have $\alpha^{(l_1-l_2)a}\beta^{b(j_{1,1}-j_{2,1})}=1$ for all $b\equiv ax(\text{mod } t)$. So if $\gamma_1\in\Gamma_{\overline{\nu}}$, we have a representation

$$\gamma_1 = l\gamma_0 + j_1\gamma_1 \text{ where } l \in \mathbb{Z},\ 0 \leqslant |j_1| < \overline{m_1}$$
$$\alpha^{la}\beta^{bj_1} = 1\ \forall\ b \equiv ax(\text{mod } t).$$

In the above expression, $(1-j_1)\gamma_1 = l\gamma_0 \in \gamma_0\mathbb{Z} \Longrightarrow \overline{m_1} \mid (1-j_1)$. And $|1-j_1| \leqslant 1+|j_1| \leqslant \overline{m_1} \Longrightarrow |1-j_1| = 0$ or $\overline{m_1}$. $1-j_1 = 0 \Longrightarrow l = 0,\ j_1 = 1$. From the above expression we then have, $\beta^b = 1\ \forall\ b \equiv ax(\text{mod } t) \Longrightarrow n = 1$. Now consider $|1-j_1| = \overline{m_1}$. If $1-j_1 = -\overline{m_1}$ then $j_1 = 1+\overline{m_1}$ which contradicts $|j_1| < \overline{m_1}$. So $1-j_1 = \overline{m_1}$, that is, $j_1 = 1-\overline{m_1}$. And $(1-j_1)\gamma_1 = \overline{m_1}\gamma_1 = l\gamma_0$. So $Q_2 = Q_1^{\overline{m_1}} - \lambda X^l$ where $\lambda \in K\backslash\{0\}$. $(\alpha^a,\beta^b)\cdot Q_2 = \beta^{b\overline{m_1}}Q_1^{\overline{m_1}} - \lambda\alpha^{al}X^l$. Since Q_2 is an eigenfunction, we have $\beta^{b\overline{m_1}} = \alpha^{al}\ \forall\ b \equiv ax(\text{mod } t)$. Again from the above expression we have, $\alpha^{al}\beta^b = \beta^{b\overline{m_1}}\ \forall\ b \equiv ax(\text{mod } t)$, as $j_1 = 1-\overline{m_1}$. Thus, $\beta^b = 1\ \forall\ b \equiv ax(\text{mod } t)$, and hence $n = 1$. So we have obtained,

$$\gamma_1 \in \Gamma_{\overline{\nu}} \Longrightarrow N = 1,\ t = 1. \tag{5.2}$$

For an element $g \in \Gamma_\nu$, let $[g]$ denote the class of g in $\frac{\Gamma_\nu}{\Gamma_{\overline{\nu}}}$. Since $\frac{\Gamma_\nu}{\Gamma_{\overline{\nu}}}$ is a finite group, $[g]$ has finite order for each $g \in \Gamma_\nu$. Let $e = [\Gamma_\nu : \Gamma_{\overline{\nu}}]$.

First we suppose $\gamma_0 \in \Gamma_{\overline{\nu}}$ and $\gamma_1 \in \Gamma_{\overline{\nu}}$. From (5.1) and (5.2) we have $M = N = t = 1$. From Proposition 1.4 we have $|H| = MNt = 1$. Thus, $MNt \mid e$.

Now we suppose $\gamma_0 \notin \Gamma_{\overline{\nu}}$ and $\gamma_1 \in \Gamma_{\overline{\nu}}$. From (5.2) we have $N = t = 1$. From Proposition 1.4 we have $|H| = MNt = M$. Let f_0 denote the order of $[\gamma_0]$. Thus $f_0\gamma_0 \in \Gamma_{\overline{\nu}}$. We thus have a representation

$$\begin{aligned} f_0\gamma_0 &= (l_1\gamma_0 + \sum_{k=1}^{r} h_{1,k}\gamma_k) - (l_2\gamma_0 + \sum_{k=1}^{r} h_{2,k}\gamma_k) \\ &= (l_1 - l_2)\gamma_0 + \sum_{k=1}^{r}(h_{1,k} - h_{2,k})\gamma_k \end{aligned}$$

where $l_1\gamma_0 + \sum_{k=1}^{r} h_{1,k}\gamma_k \in S^{A_\mathfrak{n}}(\nu)$, and $l_2\gamma_0 + \sum_{k=1}^{r} h_{2,k}\gamma_k \in S^{A_\mathfrak{n}}(\nu)$. Thus $l_1, l_2 \in \mathbb{N}$, $r \in \mathbb{N}$ and $0 \leqslant h_{1,k}, h_{2,k} < \overline{m_k}\ \forall\ k = 1,\cdots,r$. So, $|h_{1,k} - h_{2,k}| < \overline{m_k}\ \forall\ k = 1,\cdots,r$. With the same arguments as above, we have $h_{1,k} = h_{2,k}\ \forall\ k = 1,\cdots,r$. Thus $f_0\gamma_0 = (l_1-l_2)\gamma_0 \Longrightarrow f_0 = l_1 - l_2$. And, for all $b \equiv ax(\text{mod } t)$,

$$\alpha^{l_1 a}\beta^{b\sum_{k=1}^{r}[h_{1,k}d(k)]} = 1 = \alpha^{l_2 a}\beta^{b\sum_{k=1}^{r}[h_{2,k}d(k)]}.$$

So, $\alpha^{(l_1-l_2)} = \alpha^{f_0} = 1$, hence $m = Mt \mid f_0 \Longrightarrow Mt \mid e$. Thus $MNt \mid e$ as $MNt = M$.

Now we suppose $\gamma_0 \in \Gamma_{\overline{\nu}}$ and $\gamma_1 \notin \Gamma_{\overline{\nu}}$. From (5.1) we have $M = t = 1$. $|H| = MNt = N$. Let f_1 denote the order of $[\gamma_1]$, that is $f_1\gamma_1 \in \Gamma_{\overline{\nu}}$. We have a representation,

$$f_1\gamma_1=(l_1\gamma_0+\sum_{k=1}^{r}j_{1,k}\gamma_k)-(l_2\gamma_0+\sum_{k=1}^{r}j_{2,k}\gamma_k)=(l_1-l_2)\gamma_0+\sum_{k=1}^{r}(j_{1,k}-j_{2,k})\gamma_k$$

where $l_1\gamma_0+\sum_{k=1}^{r}j_{1,k}\gamma_k \in S^{A_\mathfrak{n}}(\nu)$, and $l_2\gamma_0+\sum_{k=1}^{r}j_{2,k}\gamma_k \in S^{A_\mathfrak{n}}(\nu)$. So, $l_1,l_2 \in \mathbb{N}$, $r \in \mathbb{N}$ and $0 \leqslant j_{1,k},j_{2,k} < \overline{m_k}$ $\forall$ $k=1,\cdots,r$. So, $|j_{1,k}-j_{2,k}| < \overline{m_k}$ $\forall$ $k=1,\cdots,r$. With the same arguments as above, we have $j_{1,k}=j_{2,k}$ $\forall$ $k=2,\cdots,r$. So in the above representation, we have $f_1\gamma_1=(l_1-l_2)\gamma_0+(j_{1,1}-j_{2,1})\gamma_1$ where $0\leqslant |j_{1,1}-j_{2,1}|<\overline{m_1}$. Again, $\forall$ $b\equiv ax(\text{mod } t)$ we have

$$\alpha^{l_1a}\beta^{b\sum_{k=1}^{r}[j_{1,k}d(k)]}=1=\alpha^{l_2a}\beta^{b\sum_{k=1}^{r}[j_{2,k}d(k)]}.$$

Since $d(1)=1$ and $j_{1,k}=j_{2,k}$ $\forall$ $k=2,\cdots,r$, we have $\alpha^{(l_1-l_2)a}\beta^{b(j_{1,1}-j_{2,1})}=1$ for all $b\equiv ax(\text{mod } t)$. So we have a representation,

$$f_1\gamma_1=l\gamma_0+j_1\gamma_1 \text{ where } l\in\mathbb{Z},\ 0\leqslant|j_1|<\overline{m_1}$$
$$\alpha^{la}\beta^{bj_1}=1\ \forall\ b\equiv ax(\text{mod } t).$$

$(f_1-j_1)\gamma_1=l\gamma_0 \implies \overline{m_1}\mid(f_1-j_1)$. Let $f_1-j_1=c\overline{m_1}$ where $c\in\mathbb{Z}$. Let $\overline{m_1}\gamma_1=s\gamma_0$ where $s\in\mathbb{Z}_{>0}$. Thus $f_1\gamma_1=cs\gamma_0+j_1\gamma_1\implies l\gamma_0=cs\gamma_0$. Thus $l=cs$. Since $\overline{m_1}\gamma_1=s\gamma_0$, we have $Q_2=Q_1^{\overline{m_1}}-\lambda X^s$ where $\lambda\in K\backslash\{0\}$. $(\alpha^a,\beta^b)\cdot Q_2=\beta^{b\overline{m_1}}Q_1^{\overline{m_1}}-\lambda\alpha^{as}X^s$. Since Q_2 is an eigenfunction we have, $\beta^{b\overline{m_1}}=\alpha^{as}$ $\forall$ $b\equiv ax(\text{mod } t)$. Again, from the above expression of $f_1\gamma_1$, we have

$$\alpha^{la}\beta^{b(f_1-c\overline{m_1})}=1\ \forall\ b\equiv ax(\text{mod } t)$$
$$\implies \alpha^{csa}\beta^{bf_1}=\beta^{bc\overline{m_1}}\ \forall\ b\equiv ax(\text{mod } t) \text{ as } l=cs$$
$$\implies \beta^{bf_1}=1\ \forall\ b\equiv ax(\text{mod } t)\implies n=Nt\mid f_1\implies Nt\mid e.$$

Thus we have obtained, $MNt\mid e$ as $MNt=N$.

Now we consider the final case, $\gamma_0\notin\Gamma_{\overline{\nu}}$ and $\gamma_1\notin\Gamma_{\overline{\nu}}$. Let f_0 denote the order of $[\gamma_0]$ and f_1 denote the order of $[\gamma_1]$ in $\frac{\Gamma_\nu}{\Gamma_{\overline{\nu}}}$. With the same arguments as before, we obtain $Mt\mid f_0$ and $Nt\mid f_1$. Thus we have $Mt\mid e$ and $Nt\mid e$. Now $(Mt,Nt)=t$. So the lowest common multiple of Mt and Nt is $\frac{MtNt}{t}=MNt$. Thus, $MNt\mid e$.

Now, $K(X,Y)$ is a Galois extension of $Q(A)$ with Galois group H (Proposition 1.1.1, [[2]]). Thus $[K(X,Y):Q(A)]=|H|=MNt$ from Proposition 1.4. Let $\nu=\nu_1,\nu_2,\cdots,\nu_r$ be all the distinct extensions of $\overline{\nu}$ to $K(X,Y)$. Then (§12, Theorem 24, Corollary, [[16]]),

$$efr=[K(X,Y):Q(A)]=MNt.$$

Since $MNt\mid e$, we have $e=MNt$, $r=1$. So ν is the unique extension of $\overline{\nu}$ to $K(X,Y)$. Thus $\overline{\nu}$ does not split in $R_\mathfrak{m}$. □

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