VALUED DIFFERENCE FIELDS AS MODULES OVER TWISTED POLYNOMIAL RINGS

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THESIS

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Chapter 1

Introduction

1.1 Overview

Suppose k is a field of characteristic $p > 0$ such that $[\mathrm{k} : \mathrm{k}^p]$ is finite. Let $K = \mathrm{k}((t))$ be the field of Laurent series over k. The field K is a valued field with respect to the valuation v given by

$$v(f) = \min\{\, i \in \mathbb{Z} \mid f_i \neq 0 \,\}$$

for $f = \sum_{i\in\mathbb{Z}} f_i t^i \in K$, $f \neq 0$. The thesis studies the elementary theory of K as a K-module equipped with the Frobenius map $\lambda \mapsto \lambda^p$ and with the valuation ring $V_0 = \mathrm{k}[[t]]$ as a distinguished subgroup. This study is meant as a step towards determining the elementary theory of K as a valued field, which is arguably the main problem left open by the famous papers of Ax-Kochen [AK1], [AK2] and Ershov [E1], [E2] in the sixties on the elementary theory of local fields of characteristic zero.

Let $R = K[\Phi]$ be the ring of twisted polynomials in the variable Φ over K determined by the relations $\Phi\lambda = \lambda^p\Phi$ for $\lambda \in K$. We make K into a left R-module by $\lambda \cdot \mu := \lambda\mu$ for all $\lambda \in K$ and $\Phi \cdot \mu := \mu^p$.

Let L be the language that has for every element r of R a unary function symbol $r\cdot$ and for every $i \in \mathbb{Z}$ a unary predicate V_i. We make K into an L-structure by interpreting $r\cdot$ as the action of r on K and V_i as the subgroup of elements with valuation greater than or equal to i. A main aim of the thesis is to determine the first order L-theory of this structure K.

In the model theory of modules, so-called positive primitive formulas (short: pp-formulas) play an important role. A pp-formula of L is an L-formula $\alpha(x_1, \dots, x_m)$ of the form

$$\exists y_1, \dots, y_n \alpha'(x_1, \dots, x_m, y_1, \dots, y_n)$$

where $x_1, \dots, x_m, y_1, \dots, y_n$ are distinct variables and α' is a conjunction of atomic formulas. The solution set in the L-structure K of such a pp-formula $\alpha(x_1, \dots, x_m)$ is a subgroup of the additive product group K^m, and is called a **pp-definable subset** (or **pp-definable subgroup**) of K^m. (Since R is not commutative, a pp-definable subgroup of K^m is not always an R-submodule of the product module K^m.)

A general result about the model theory of modules (see [Ba], [Mo]) stated in Theorem 5.13 implies that every subset of K^m that is definable in the L-structure K is a boolean combination of pp-definable subsets of K^m. Also, the first order theory of the L-structure K is determined by the indices $|\alpha(K)/(\alpha(K) \cap \beta(K))| \in \mathbb{N} \cup \{\infty\}$ for all pp-formulas $\alpha(x), \beta(x)$. Here, and throughout the thesis, "definable" means "definable without parameters" unless indicated otherwise.

One goal of the thesis is to obtain a procedure to compute the quantity $|B/A|$ for pp-definable subgroups A and B of K with $A \subseteq B$. The dissertation will also contain results on the structure of pp-definable subsets of K^m. For example, such sets are closed with respect to the valuation topology (see Corollary 7.28). Another result about the model theory of the L-structure K is the following: If k is model-complete as a module over the subring $\mathrm{k}[\Phi]$ of $K[\Phi]$, then the L-structure K is model-complete. In particular, the L-structure K is model-complete, if k is finite or algebraically closed.

As the main step, results on the "small" and "large" structure of a pp-definable set are obtained. Here the "small" structure of a pp-definable set is the structure of the set after intersecting with a sufficiently small valuation ball V_i (i.e. for some large enough i). The "large" structure of a pp-definable set is the structure of the set after adding a sufficiently large valuation ball V_i (i.e. for some small i), which results in blurring the small details.

More generally, one can consider an arbitrary valued field (K, v) instead of $(\mathrm{k}((t)), v)$ and replace the Frobenius map $\lambda \mapsto \lambda^p$ in the commutation rule $\Phi\lambda = \lambda^p\Phi$ by an arbitrary self-embedding ϕ of the valued field (K, v) to obtain a ring $K[\Phi]$ with $\Phi\lambda = \phi(\lambda)\Phi$ for $\lambda \in K$. Many considerations go through in this more general setting with the following additional assumptions, which are true for $K = \mathrm{k}((t))$ and ϕ the Frobenius map:

1. The value group $\Gamma = v(K \setminus \{0\})$ is non-trivial and ϕ increases positive valuations (see Definition 6.12 and Assumption 6.18). This generalizes the Frobenius case $v(\lambda^p) = pv(\lambda)$.

2. K has finite dimension over its subfield $\phi(K)$, and there exists a weakly valuation independent basis of K over $\phi(K)$ (see Definition 6.3). Such a basis allows one to obtain upper estimates on the valuation of a linear combination of basis elements in terms of the coefficients. For $K = \mathrm{k}((t))$ and B a basis of k over k^p, the set $\{ bt^i \mid b \in B, 0 \leq i < p \}$ is such a basis.

3. Certain pp-definable subsets S of K satisfy a weak optimal approximation property (see Definition 7.2 and Assumption 7.15). This means that for every point $a \in K$, there is among elements of S one that is closest to a in some weak sense.

   In the situation where the value group of K is $\mathbb{Z}$, for example, if $K = \mathrm{k}((t))$, this is trivially satisfied. But it also holds in all maximally valued fields (see Proposition 7.13). This assumption is used to obtain a nice "large"-asymptotic notion of size for pp-definable sets (see Lemma 7.14, Definition 7.16).

4. K is linear ϕ-henselian (see Definition 7.20). For $K = \mathrm{k}((t))$, this holds, because $\mathrm{k}((t))$ is henselian. This assumption is used to obtain a nice "small" structure of the pp-definable sets

(see Proposition 7.27).

Every finite system of homogeneous linear equations in the finite set of variables X with coefficients in $K[\Phi]$ can be written as $M \cdot x = 0_I$ where M is a matrix over $K[\Phi]$ with some finite row index set I and column index set X, where 0_I is a column vector of zeros with row index set I, and where x is a column vector whose entries are the variables in X. Since pp-formulas are closely related to such systems of linear equations, a study of matrices over $K[\Phi]$ seems natural.

As preparation, some known properties of the ring $K[\Phi]$ are summarized in Chapter 2. Most important here is the Division Lemma 2.6, which is a non-commutative analogue to the Euclidean property for a polynomial ring over a field. In Chapter 3, we study matrices over the ring $K[\Phi]$. The aim is to obtain ways to transform an arbitrary matrix into a nicer form using elementary row and column operations. For example, using elementary row operations, one can always transform a matrix into an upper triangular form (see Definition 3.7). Lemma 3.15 will be crucial in showing that in the "small", pp-definable sets are just defined by systems of homogeneous linear equations (so no projection is necessary) after applying some definable bijection (see Proposition 7.27). Proposition 3.18 will be crucial in showing that in the "large", pp-definable sets are just given as images under a term map, so no equations are necessary in some sense (see Proposition 7.5).

In Chapter 5, the $K[\Phi]$-module structure of K is formally defined and some general results in the model theory of modules summarized. Chapter 6 deals with the basic valuation estimates needed to analyze the "small" and "large" structure of pp-definable sets. Most crucial here is Proposition 6.26, which provides a lower estimate for the valuation of $\lambda_i \in K$ in terms $v(\sum_i f_i \cdot \lambda_i)$ when the $f_i \in K[\Phi]$ are strongly independent (see Definition 3.16). This proposition is used later in the proof of Lemma 7.11 and Proposition 7.13.

Chapter 7 deals with the "large" (see Section 7.1) and "small" (see Section 7.2) structure of pp-definable sets. In both subsections, first a result is obtained that shows that the asymptotic structure of pp-definable sets is less complicated: In the "large" case, one can essentially dispense with equational constraints (see Proposition 7.5) and in the "small" case, one can essentially dispense with projections (existential quantifiers) (see Proposition 7.27). Using this simplified structure, one can in both cases define an asymptotic measure of the size for a pp-definable set, denoted by $\dim_0$ (dimension in the "small" setting, i.e. for large positive valuations) and $\dim_\infty$ (dimension in the "large" setting, i.e. for large negative valuations) respectively. These dimensions have values in

$$\left\{ \frac{i}{j} \;\middle|\; i \in \mathbb{N}, j \in \{\, [K : \phi(K)]^m \mid m \in \mathbb{N} \,\} \right\}.$$

The dimensions are not definable invariants, but depend on the ambient space K^n. Let $C \subseteq A \subseteq K^n$ be pp-definable sets. Then $\dim_\infty C \leq \dim_\infty A$ and $\dim_0 C \leq \dim_0 A$, $\dim_\infty C = \dim_\infty A$ if and only if there exists some (possibly large) valuation ball B_0 in K^n such that $A + B_0 = C + B_0$, and $\dim_0 C = \dim_0 A$ if and only if there exists some (possibly small) valuation ball B_1 in K^n such that $A \cap B_1 = C \cap B_0$. If $\dim_\infty C < \dim_\infty A$ or $\dim_0 C < \dim_0 A$, then $|A/C|$ is infinite. Otherwise, one can reduce the computation of $|A/C|$ to the computation of $|A'/C'|$ for some pp-definable sets A',

C' in some simpler factor module (see Result 8.11).

Chapter 4 provides some results about finitely generated free modules over $K[\Phi]$ and submodules of such modules. These results are needed for the definition of $\dim_\infty$ and $\dim_0$. It turns out that for $\dim_\infty$ the free right $K[\Phi]$-modules play a role (see Section 4.1), and for $\dim_0$ the free left $K[\Phi]$-modules play a role (see Section 4.2).

In Chapter 8, the asymptotic description of pp-definable sets is used to reduce the computation of pp-indices to certain factor modules of the valuation ring of K, which have in general a simpler structure than the module K. We also obtain a statement that model-completeness of all these factor modules implies model completeness of the K, if the valuation ring is pp-definable in K. In Section 8.2, we obtain a corresponding reduction to statements about the residue field k of K, if the value group Γ of K is $\mathbb{Z}$ and k is embedded in K with $\phi(\mathrm{k}) \subseteq \mathrm{k}$.

1.2 Notations and Conventions

We let $\mathbb{N}$ denote the set of natural numbers including 0. If not otherwise stated, the letters i, j, k, m and n denote elements of $\mathbb{N}$. For example, the statement "$i < n$" means $i, n \in \mathbb{N}$ and $i < n$. The set $\{ i \mid i < n \}$ is sometimes simply denoted by n.

For sets S and I the symbol S^I stands for the set of maps from I to S. An element of S^I is also called a tuple indexed by I over S. Saying that $a = (a_i)_{i \in I}$ is a tuple over S means that $a \in S^I$ and $a(i) = a_i$ for all $i \in I$. Such a tuple is said to be **finite**, if its index set I is finite. Let S^n denote the set $S^{\{ i \mid i<n \}}$. We identify this set with the n-fold Cartesian product $S \times \cdots \times S$. For a function $f : A \longrightarrow B$, its restriction to $A_0 \subseteq A$ is denoted by $f{\restriction}_{A_0}$.

Throughout we assume that languages and formulas are first-order and one-sorted.

Chapter 2

Rings

This section summarizes properties of twisted polynomial rings over fields and of some auxiliary rings. In parts, the treatment is more general than needed for later applications. “Ring” always means “not necessarily commutative ring with 1”.

2.1 Definition. Let R be a ring. A **unit** in R is an element of R with a two-sided multiplicative inverse, and $R^\times$ denotes the set of units of R. An element $r \in R$ is called **right regular**, if $s = 0$ for every $s \in R$ with $rs = 0$. An element $r \in R$ is called a **left zero-divisor**, if $r \neq 0$ and $rs = 0$ for some non-zero $s \in R$. There are corresponding definitions with right and left interchanged. We call R a **domain**, if R has no left and no right zero-divisors, and $0 \neq 1$ in R.

2.2 Remark. Let R be a ring.

1. An element in R is a left zero-divisor if and only if it is non-zero and not right regular.
2. The units in R form a group with respect to the multiplication.
3. Suppose R is a domain. If $r \in R$ has a left (or right) inverse, then this inverse is actually a two-sided inverse.

2.3 Definition. Let R be a ring and ϕ be a ring endomorphism of R. Let

$$R[[\Phi]] := R^{\mathbb{N}}$$

be the set of sequences over R. Define the operations

$$+\,,\,\cdot\,:\; R[[\Phi]] \times R[[\Phi]] \longrightarrow R[[\Phi]]$$

by

$$(f+g)(i) := f(i) + g(i)$$

and

$$(f \cdot g)(i) := \sum_{j \leq i} f(j)\phi^j(g(i-j))\,.$$

An element f is also denoted by $\sum_{i\in\mathbb{N}} f(i)\Phi^i$. These operations make $R[[\Phi]]$ a ring with $1 = 1\Phi^0$; it is called the ring of **(left) twisted power series over** R **with respect to** ϕ. Let

$$R[\Phi] := \{\, f \in R[[\Phi]] \mid f(i) = 0 \text{ for all but finitely many } i \in \mathbb{N} \,\} \,.$$

This is a subring of $R[[\Phi]]$, called the **ring of twisted polynomials over** R **with respect to** ϕ. For f in $R[\Phi] \setminus \{\, 0 \,\}$, define the **degree of** f by

$$\deg(f) := \max \{\, i \in \mathbb{N} \mid f(i) \neq 0 \,\}$$

and call $f(\deg(f))$ the **leading coefficient of** f. Furthermore, set $\deg(0) := -\infty$. For f in $R[[\Phi]] \setminus \{\, 0 \,\}$, define the **order** (or **lower degree**) **of** f by

$$\operatorname{ldeg}(f) := \min \{\, i \in \mathbb{N} \mid f(i) \neq 0 \,\}$$

and call $f(\operatorname{ldeg}(f))$ the **trailing coefficient of** f. Furthermore, set $\operatorname{ldeg}(0) := \infty$. We write Φ for the element $1\Phi^1 \in R[[\Phi]]$.

2.4 Remark. Consider the situation of the previous definition.

1. Usually, the ring endomorphism ϕ is given and so does not appear in the notation for the twisted power series and twisted polynomial ring.

2. R becomes a subring of $R[[\Phi]]$ via the embedding $r \mapsto r\Phi^0$. Note that then $\Phi r = \phi(r)\Phi$ for $r \in R$.

3. $R[\Phi]$ satisfies the following universal property: For every ring homomorphism $\rho : R \longrightarrow S$ and every element $s \in S$ such that $s\rho(r) = \rho(\phi(r))s$ for all $r \in R$, there exists a unique ring homomorphism $\hat{\rho} : R[\Phi] \longrightarrow S$ that extends ρ and maps Φ to s.

4. Suppose I is a two-sided ideal of R such that $\phi(I) \subseteq I$. Then ϕ induces a ring endomorphism $\bar{\phi}$ of R/I by $\bar{\phi}(r/I) = \phi(r)/I$. Let $(R/I)[[\Phi]]$ denote the twisted power series ring with respect to the endomorphism $\bar{\phi}$. Let $I[[\Phi]] := \left\{\, \sum_{i\in\mathbb{N}} f(i)\Phi^i \;\middle|\; f \in I^{\mathbb{N}} \,\right\}$. Then $I[[\Phi]]$ is a two-sided ideal of $R[[\Phi]]$ and $R[[\Phi]]/I[[\Phi]]$ is canonically isomorphic to $(R/I)[[\Phi]]$. A similar statement holds for twisted polynomial rings instead of twisted power series rings.

2.5 Lemma.

1. *Let* $f, g \in R[\Phi]$. *Then*
$$\deg(f - g) \leq \max \{\, \deg(f), \deg(g) \,\}$$
and
$$\deg(fg) \leq \deg(f) + \deg(g) \,.$$
If $g \neq 0$ *and the leading coefficient of* g *is a unit in* R*, then the second inequality becomes an equality.*

2. *Let $f, g \in R[[\Phi]]$. Then*
$$\operatorname{ldeg}(f-g) \geq \min\{\operatorname{ldeg}(f), \operatorname{ldeg}(g)\}$$
and
$$\operatorname{ldeg}(fg) \geq \operatorname{ldeg}(f) + \operatorname{ldeg}(g)\,.$$
If $g \neq 0$ and the trailing coefficient of g is a unit in R, then the second inequality becomes an equality.

3. *Suppose R is a domain and ϕ is injective. Then the equality $\deg(fg) = \deg(f) + \deg(g)$ holds in $R[\Phi]$ and $\operatorname{ldeg}(fg) = \operatorname{ldeg}(f) + \operatorname{ldeg}(g)$ holds in $R[[\Phi]]$. Furthermore, $R[[\Phi]]$ and $R[\Phi]$ are domains.*

The proof is obvious.

2.6 Lemma (Division lemma). *Let $f, g \in R[\Phi]$ such that $g \neq 0$ and the leading coefficient of g is a unit. Then there exist unique $q, r \in R[\Phi]$ such that $f = qg + r$ and $\deg(r) < \deg(g)$.*

Proof. Existence: We proceed by induction on the degree of f. If $\deg(f) < \deg(g)$, one can choose $q = 0$ and $r = f$. So suppose $f = \sum_{i \leq m} f_i \Phi^i$ and $g = \sum_{i \leq n} g_i \Phi^i$ with $f_m \neq 0$ and $g_n \neq 0$ and $m \geq n$. Let $q_0 := f_m \Phi^{m-n} g_n^{-1}$. Then $f' := f - q_0 g$ has degree less than m, so by the induction hypothesis, there exist $q', r' \in R[\Phi]$ with $f' = q'g + r'$ and $\deg(r') < \deg(g)$. Now let $q = q_0 + q'$ and $r = r'$. Then $f = q_0 g + f' = (q_0 + q')g + r$, and $\deg(r) < \deg(g)$. This proves existence.

Uniqueness: It suffices to show that for $q, r \in R[\Phi]$ the conditions $qg + r = 0$ and $\deg(r) < \deg(g)$ imply $q = 0$ and $r = 0$. But by the previous lemma, since the leading coefficient of g is a unit, $\deg(g) > \deg(r) = \deg(qg) = \deg(q) + \deg(g)$, which implies $q = 0$. ☐

2.7 Remark.

1. $d(f, g) := 2^{-\operatorname{ldeg}(f-g)}$ defines a complete metric on $R[[\Phi]]$.

2. If $f \in R[[\Phi]]$ with $\operatorname{ldeg}(f) > 0$, then the sequence $(\sum_{i=0}^{n} f^i)_n$ converges in the above metric. The limit denoted by $\sum_{i \in \mathbb{N}} f^i$ is a two-sided inverse of $1 - f$.

3. $f = \sum_{i \in \mathbb{N}} f_i \Phi^i$ is a unit in $R[[\Phi]]$ if and only if f_0 is a unit in R.

4. For a complete commutative local ring R and a local ring endomorphism ϕ of R, the following one-sided analogue of the Weierstrass division theorem holds in $R[[\Phi]]$: Let $\mathfrak{m}$ be the maximal ideal of R. Let $f = \sum_{i \in \mathbb{N}} f(i) \Phi^i \in R[[\Phi]]$ and $d \in \mathbb{N}$ such that $f(i) \in \mathfrak{m}$ for $i < d$ and $f(d)$ is a unit in R. Then for each $g \in R[[\Phi]]$ there exist unique $q \in R[[\Phi]]$ and $r \in R[\Phi]$ such that $g = qf + r$ and $\deg(r) < d$. (The statement is not true in general, if qf is replaced by fq.)

   This version of the Weierstrass division is not used in the thesis.

2.8 Definition. Let R be a ring and S be a **multiplicative subset** of R, i.e. S contains 1 and is closed under multiplication. The set S is said to be a **left denominator set** if the following two conditions hold:

1. **left Ore condition**: $Rs \cap Sr \neq \emptyset$ for all $s \in S$ and $r \in R$ (equivalently $Rs \cap Sr \neq \emptyset$ for all $s \in S$ and $r \in R \setminus \{0\}$).

2. If $rs = 0$ for $r \in R$ and $s \in S$, then there exists $s' \in S$ such that $s'r = 0$.

2.9 Remark (compare [Row], section 3.1). Let the multiplicative subset S of R be a left denominator set. Then there exist a ring T and a ring homomorphism $\rho : R \longrightarrow T$ such that $\rho(s)$ is a unit for all $s \in S$ and ρ is universal with this property. The universality means that for every ring homomorphism $\rho' : R \longrightarrow T'$ such that $\rho'(s)$ is a unit for all $s \in S$ there exists a unique ring homomorphism $\tau : T \longrightarrow T'$ with $\rho' = \tau\rho$. It follows that, as an R-ring, T is unique up to unique isomorphism. Every element in T can be written as $\rho(s)^{-1}\rho(r)$ for $r \in R$ and $s \in S$. The ring T is called the **ring of left fractions of R with respect to S** and is denoted by $S^{-1}R$.

For every left ideal I of R, the set $S^{-1}I = \{ \rho(s)^{-1}\rho(r) \mid s \in S, r \in I \}$ is a left ideal of $S^{-1}R$; every left ideal of $S^{-1}R$ is of this form.

If in addition every element in S is right regular, then ρ is an embedding. In particular, if R is a domain, then for every multiplicative subset S of $R \setminus \{0\}$ that satisfies the left Ore condition the canonical map $R \longrightarrow S^{-1}R$ is an embedding, and we shall identify R with a subring of $S^{-1}R$ via this embedding.

2.10 Lemma/Definition. Let K be a field and ϕ be a self-embedding of K.

1. $K[[\Phi]]$ and $K[\Phi]$ are domains. The group of units of $K[\Phi]$ is $K^\times = K \setminus \{0\}$.

2. Every left ideal of $K[\Phi]$ is a principal left ideal. In particular $K[\Phi]$ is left Noetherian.

3. The left ideal of $K[\Phi]$ generated by Φ^n is a two-sided ideal. The two-sided ideal $\mathfrak{m} := K[\Phi]\Phi$ is maximal as a right and as a left ideal; the map $K \longrightarrow K[\Phi]/\mathfrak{m}$, $\lambda \mapsto \lambda + \mathfrak{m}$ is a ring isomorphism.

4. The sets $S_0 = K[\Phi] \setminus \{0\}$ and $S_\mathfrak{m} = K[\Phi] \setminus \mathfrak{m}$ are left denominator sets of $K[\Phi]$. The ring of left fractions $(S_0)^{-1}K[\Phi]$ is denoted by $Q(K[\Phi])$ and the ring of left fractions $(S_\mathfrak{m})^{-1}K[\Phi]$ by $K[\Phi]_\mathfrak{m}$.

5. The left ideal of $K[\Phi]_\mathfrak{m}$ generated by Φ is a two-sided ideal and it is a largest proper left and a largest proper right ideal.

Proof. 1. This follows from Lemma 2.5.

2. Let I be a left ideal of $K[\Phi]$. If $I = \{0\}$, then $I = K[\Phi]0$. Otherwise, there exists an element $g \in I \setminus \{0\}$ with minimal degree. Now let $f \in I$. By the division lemma, there exist $q, r \in K[\Phi]$ such that $f = qg + r$ and $\deg(r) < \deg(g)$. Since $f, g \in I$, and I is a left ideal, we have $r \in I$, so by the choice of g it follows that r is 0. Therefore $f \in K[\Phi]g$. This shows $I = K[\Phi]g$.

3. Since for every element $f = \sum_{i<m} f_i \Phi^i \in K[\Phi]$ we have $\Phi^n f = (\sum_{i<m} \phi^n(f_i)\Phi^i)\Phi^n$, the left ideal $K[\Phi]\Phi^n$ is also a right ideal. It is easy to see that the inclusion $K \longrightarrow K[\Phi]$ induces a ring isomorphism $K \simeq K[\Phi]/\mathfrak{m}$, which yields also the maximality of $\mathfrak{m}$ as a left and as a right ideal.

4. Since $K[\Phi]/0$ and $K[\Phi]/\mathfrak{m}$ are domains, the sets S_0 and $S_\mathfrak{m}$ are multiplicative. Since they don't contain 0 and $K[\Phi]$ is a domain, they satisfy the second condition of being a left denominator set.

   We first show the left Ore condition for S_0. Let $r \in K[\Phi]$ and $s \in S_0$. We have to show that there exist $f \in S_0$ and $g \in K[\Phi]$ such that $fr = gs$. If $r = 0$, then one can choose $f = 1$ and $g = 0$, so assume now $r \neq 0$.

   Consider $K[\Phi]_{\leq i} := \{ h \in K[\Phi] \mid \deg h \leq i \}$ as K-vector space where the action is given by left multiplication viewing K as a subset of $K[\Phi]$. We have $\dim_K K[\Phi]_{\leq i} = i + 1$.

   Consider the K-linear map

   $$K[\Phi]_{\leq \deg s} \times K[\Phi]_{\leq \deg r} \longrightarrow K[\Phi]_{\leq \deg s + \deg r}\,,\ (f, g) \mapsto fr - gs\,.$$

   It maps the $(\deg s + 1 + \deg r + 1)$-dimensional K-vector space $K[\Phi]_{\leq \deg s} \times K[\Phi]_{\leq \deg r}$ into the $(\deg s + \deg r + 1)$-dimensional K-vector space $K[\Phi]_{\leq \deg s + \deg r}$. Thus, there exists a non-zero element in the kernel of this map, i.e. there exists $f \in K[\Phi]_{\leq \deg s}$ and $g \in K[\Phi]_{\leq \deg r}$ with $fr - gs = 0$, and $f \neq 0$ or $g \neq 0$. Since $K[\Phi]$ is a domain and $s \neq 0$, we can conclude that $f \neq 0$, so $f \in S_0$, and we are done.

   Next, we show the left Ore condition for $S_\mathfrak{m}$. Let $r \in K[\Phi]$ and $s \in S_\mathfrak{m}$, so $\operatorname{ldeg} s = 0$. We have to show that there exist $f \in S_m$ and $g \in K[\Phi]$ such that $fr = gs$. Again, we can assume $r \neq 0$. Applying the left Ore condition for S_0, we obtain $f' \in S_0$ and $g' \in K[\Phi]$ such that $f'r = g's$. We have $\operatorname{ldeg} g' = \operatorname{ldeg} g's = \operatorname{ldeg} f'r \geq \operatorname{ldeg} f'$. With $j := \operatorname{ldeg} f'$, we obtain $g', f' \in K[\Phi]\Phi^j$. Let $B \subseteq K$ be a basis of the $\phi^j(K)$-vector space K. The tuple $(b\Phi^j)_{b\in B}$ is a basis of the right $K[\Phi]$-module $K[\Phi]\Phi^j$. Therefore, there exist $(f_b)_{b\in B}, (g_b)_{b\in B} \in K[\Phi]^B$ with $f_b = 0, g_b = 0$ for all but finitely $b \in B$ and $\sum_{b\in B} b\Phi^j f_b = f'$, $\sum_{b\in B} b\Phi^j g_b = g'$. Since $f'r = g's$ and $(b\Phi^j)_{b\in B}$ is independent in the right $K[\Phi]$-module $K[\Phi]$, we obtain $f_b r = g_b s$ for $b \in B$. Because $\operatorname{ldeg} f' = j$, there exists $b \in B$ such that $\operatorname{ldeg} f_b = 0$, and for such b we can take $f = f_b$ and $g = g_b$.

5. Let $\mathfrak{m}'$ denote the left ideal generated by Φ in $K[\Phi]_\mathfrak{m}$. As in the proof of part 3, one shows that $\mathfrak{m}'$ is a right ideal. It is easy to see that $1 \notin \mathfrak{m}'$ and that every element in $K[\Phi]_\mathfrak{m} \setminus \mathfrak{m}'$ is a unit.

□

Similar arguments prove the next lemma.

2.11 Lemma. *Let K be a field and ϕ be a self-embedding of K.*

1. *The left ideal of $K[[\Phi]]$ generated by Φ^n is a two-sided ideal.*
2. *The set of units of $K[[\Phi]]$ is $K[[\Phi]] \setminus K[[\Phi]]\Phi$.*
3. *Every left ideal of $K[[\Phi]]$ is a principal left ideal of the form $K[[\Phi]]\Phi^n$ for some n, and every left ideal is a right ideal. In particular, $K[[\Phi]]$ is left Noetherian.*
4. *Let $\mathfrak{m} := K[[\Phi]]\Phi$. Then $\mathfrak{m}$ is a two-sided ideal of $K[[\Phi]]$ and it is a largest proper left and a largest proper right ideal.*
5. *The map $K \longrightarrow K[[\Phi]]/\mathfrak{m}\,,\ \lambda \mapsto \lambda + \mathfrak{m}$ is a ring isomorphism.*

Chapter 3

Matrix Operations

3.1 Definition. A matrix over the set R with row index set I_1 and column index set I_2 is a map from $I_1 \times I_2$ to R. The set of all such matrices is denoted by $R^{I_1\times I_2}$.

3.2 Remark/Definition. Let R be a ring and V_l be a left R-module and V_r be a right R-module. Let I_1, I_2, I_3, I be finite sets. The set V_l^I has naturally the structure of a left R-module by adding and multiplying componentwise. Then we have the usual operations of matrix multiplication

$$\begin{aligned} R^{I_1\times I_2} \times V_l^{I_2\times I_3} &\longrightarrow V_l^{I_1\times I_3}\,,\ (M,N)\mapsto MN \\ V_r^{I_1\times I_2} \times R^{I_2\times I_3} &\longrightarrow V_r^{I_1\times I_3}\,,\ (M,N)\mapsto MN \\ R^{I_1\times I_2} \times R^{I_2\times I_3} &\longrightarrow R^{I_1\times I_3}\,,\ (M,N)\mapsto MN \end{aligned}$$

given by $(MN)(i_1,i_3) = \sum_{i_2\in I_2} M(i_1,i_2)N(i_2,i_3)$. If $M_1 \in R^{I_1\times I_2}$, $M_2 \in R^{I_2\times I_3}$ and $N \in V_l^{I_3\times I_4}$, then $(M_1M_2)N = M_1(M_2N)$.

The set $\mathrm{MAT}_I(R) := R^{I\times I}$ forms a ring with the natural operations. The group of units of $\mathrm{MAT}_I(R)$ is denoted by $\mathrm{GL}_I(R)$. The identity matrix of $\mathrm{MAT}_I(R)$ is denoted by Id or Id_I to emphasize the index set I.

We call $M \in R^{I_1\times I_2}$ invertible if there exists $N \in R^{I_2\times I_1}$ such that $MN = \mathrm{Id}_{I_1}$ and $NM = \mathrm{Id}_{I_2}$.

Given disjoint sets I_0, I_1, disjoint sets J_0, J_1 and $M_i \in R^{I_i\times J_i}$ for $i = 0,1$, define $M_0 \sqcup M_1 \in R^{(I_0\cup I_1)\times(J_0\cup J_1)}$ by $(M_0 \sqcup M_1)\restriction_{I_i\times J_i} = M_i$ and $(M_0 \sqcup M_1)\restriction_{I_i\times J_{1-i}} = 0$ for $i = 0,1$.

3.3 Definition. Let I be an index set and R be a ring. For $i, i_1, i_2 \in I$ and $r \in R$, define $D_{i_1,i_2}, E_{i_1,i_2,r}, F_{i,r}$ in $\mathrm{MAT}_I(R)$ by $D_{i_1,i_2}(j_1,j_2) = \delta_{i_1,j_1}\delta_{i_2,j_2}$ for $j_1, j_2 \in I$, $E_{i_1,i_2,r} = \mathrm{Id} + rD_{i_1,i_2}$ and $F_{i,r} = \mathrm{Id} + (r-1)D_{i,i}$.

A **restricted elementary** matrix in $\mathrm{MAT}_I(R)$ is a matrix of the form $E_{i_1,i_2,r}$ with $i_1 \neq i_2$. An **elementary** matrix in $\mathrm{MAT}_I(R)$ is a restricted elementary matrix or a matrix of the form $F_{i,r}$ with r a unit in R.

3.4 Remark.

1. The elementary matrices lie in $\mathrm{GL}_I(R)$ and have inverses that are also elementary matrices: $E^{-1}_{i_1,i_2,r} = E_{i_1,i_2,-r}$ ($i_1 \neq i_2$) and $F^{-1}_{i,r} = F_{i,r^{-1}}$ (r a unit of R).

2. The result of multiplying a matrix $M \in R^{I\times J}$ by $E_{i_1,i_2,r} \in \mathrm{MAT}_I(R)$ from the left is the same as adding the i_2-row of M multiplied on the left by r to the i_1-row of M. The result of multiplying a matrix $M \in R^{I\times J}$ by $E_{j_1,j_2,r} \in \mathrm{MAT}_J(R)$ from the right is the same as adding the j_1-column of M multiplied on the right by r to the j_2-column of M.

3. An upper triangular matrix in $\mathrm{MAT}_I(R)$ with all diagonal entries equal to 1 is a product of restricted elementary matrices.

From now on, assume that K is a field, ϕ a self-embedding of K and $K[\Phi]$ the associated ring of twisted polynomials.

3.5 Remark/Definition. Let $\overline{}$ be the unique ring homomorphism $K[\Phi] \longrightarrow K$ that is the identity on K and maps Φ to 0. Let $M \in K[\Phi]^{I\times J}$. By $\overline{M}$ we denote the matrix in $K^{I\times J}$ that is obtained by applying $\overline{}$ to the entries of M.

We define $\operatorname{rank}_K M$ as the rank of the matrix $\overline{M}$ and call M **row regular**, if $\operatorname{rank}_K M$ is equal to the number of rows $|I|$ of M. Similarly, M is called **column regular**, if $\operatorname{rank}_K M$ is equal to the number of columns $|J|$ of M.

3.6 Remark. Let $M \in K[\Phi]^{I\times J}$, and $E \in \mathrm{GL}_I(K[\Phi])$. Then $\operatorname{rank}_K EM = \operatorname{rank}_K M$. In particular, being row regular and being column regular are preserved by multiplying M on the left with elements in $\mathrm{GL}_I(K[\Phi])$. The analogous statements are true for multiplication with elements in $\mathrm{GL}_J(K[\Phi])$ on the right.

3.7 Definition. Let $M \in K[\Phi]^{I\times J}$. Then M is said to be in **upper triangular form** with respect to $I_0 \subseteq I$, an injection $\iota : I_0 \longrightarrow J$ and a total order $\leq$ on I_0, if

1. M is zero on $((I \setminus I_0) \times J) \cup \{\, (i_1, \iota(i_2)) \mid i_1, i_2 \in I_0, i_1 > i_2 \,\}$;

2. for all $i_1, i_2 \in I_0$ with $i_1 \leq i_2$,
$$\operatorname{ldeg} M(i_1, \iota(i_1)) \leq \operatorname{ldeg} M(i_2, \iota(i_2)) < \infty\,;$$

3. for all $i \in I_0$ and $j \in J$,
$$\operatorname{ldeg} M(i, \iota(i)) \leq \operatorname{ldeg} M(i, j)\,.$$

Note that, if M has some upper triangular form, I_0 is uniquely determined as the set of indices of non-zero rows. We call $M{\restriction}_{I_0\times J}$ the **non-zero part** of M. If we say M **is in upper triangular form**, we fix ι and $\leq$ as above.

Here is a picture of a matrix $M \in K[\Phi]^{I\times J}$ that is in upper triangular form with respect to $I_0 = \{\, i_1, \ldots, i_m \,\}$, $\leq$ given by $i_1 < i_2 < \ldots < i_m$, and some injection $\iota : I_0 \longrightarrow J$. Note that $J \setminus \iota(I_0) = \{\, j_1, \ldots, j_n \,\}$ in this picture:

$$\begin{pmatrix}
M_{(i_1,\iota(i_1))} & \cdots & M_{(i_1,\iota(i_m))} & M_{(i_1,j_1)} & \cdots & M_{(i_1,j_n)} \\
0 & \ddots & \vdots & \vdots & & \vdots \\
\vdots & \ddots & & & & \\
0 & \cdots \; 0 & M_{(i_m,\iota(i_m))} & M_{(i_m,j_1)} & \cdots & M_{(i_m,j_n)} \\
0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & & & & & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0
\end{pmatrix}$$

3.8 Remark. Suppose $M \in K[\Phi]^{I\times J}$ is in upper triangular form with respect to $\iota : I_0 \longrightarrow I$ and $\leq$ as above. Then $\operatorname{rank}_K M = |\{\, i \in I_0 \mid \operatorname{ldeg} M(i, \iota(i)) = 0 \,\}|$ and the non-zero part of M is row regular if and only if $\operatorname{ldeg} M(i, \iota(i)) = 0$ for all $i \in I_0$.

3.9 Lemma. *Let I, J be finite sets, $M \in K[\Phi]^{I\times J}$. Then there exists a (possibly empty) product E of restricted elementary matrices in $\mathrm{MAT}_I(K[\Phi])$ such that EM is in upper triangular form.*

Proof. We start the proof by noting that for each $j \in J$ the value $\min_{i\in I} \operatorname{ldeg} M(i,j)$ is determined by the left ideal of $K[\Phi]$ generated by the elements in the j-column, so $\min_{i\in I} \operatorname{ldeg} M(i,j) = \min_{i\in I} \operatorname{ldeg}(EM)(i,j)$ for every product E of elementary matrices.

The proof is by induction on the cardinality of J. If all entries of M are zero, then M is already in upper triangular form with respect to $I_0 = \emptyset$. Otherwise, pick a column with index $j_0 \in J$ such that $\operatorname{ldeg} M(i_0, j_0) = \min_{i\in I, j\in J} \operatorname{ldeg} M(i,j)$ for some $i_0 \in I$. If for some $i_1 \in I \setminus \{\, i_0 \,\}$ the entry $M(i_1, j_0)$ is non-zero, one can by the division lemma find an element $r \in K[\Phi]$ such that for $E_1 := E_{i_0,i_1,r}$ (in case $\deg M(i_1, j_0) \leq \deg M(i_0, j_0)$) or $E_1 := E_{i_1,i_0,r}$ (in case $\deg M(i_0, j_0) \leq \deg M(i_1, j_0)$) the sum of degrees of non-zero entries in the j_0-column of E_1M is smaller than the corresponding sum for the matrix M. By induction on this sum, one can find a product of restricted elementary matrices E_2 such that the j_0-column of E_2M has exactly one non-zero entry, say in the i_2-row. By the note at the start of the proof, we have $\operatorname{ldeg}(E_2M)(i_2, j_0) = \min_{i\in I, j\in J} \operatorname{ldeg}(E_2M)(i,j)$. By applying the induction hypothesis to the sub-matrix M' of E_2M with row index set $I' = I \setminus \{\, i_2 \,\}$ and column index set $J' = J \setminus \{\, j_0 \,\}$, we find a product E' of restricted elementary matrices in $\mathrm{MAT}_{I'}(K[\Phi])$ and data $I'_0 \subseteq I'$, $\iota' : I'_0 \longrightarrow J'$ and $\leq'$ on I'_0 such that $E'M'$ is in upper triangular form with respect to the data. Then $E^* \in \mathrm{MAT}_I(K[\Phi])$ defined by $E^*(i,j) = E(i,j)$ for $i, j \in I'$, $E^*(i_2, i_2) = 1$ and $E^*(i, i_2) = 0, E^*(i_2, i) = 0$ for $i \in I'$ is a product of restricted elementary matrices in $\mathrm{MAT}_I(K[\Phi])$. Set $E := E^*E_2$. Then EM is in upper triangular form with respect to $I_0 := I'_0 \cup \{\, i_2 \,\}$, $\iota := \iota' \cup \{\, (i_2, j_0) \,\}$ and the order $\leq$ on I_0 obtained by extending ι' with i_2 as the new smallest element. ☐

3.10 Corollary. *The group of units of* $\mathrm{MAT}_I(K[\Phi])$ *is generated as a semigroup by the elementary matrices.*

For the rest of the chapter, assume that K is a finite dimensional vector space over $\phi(K)$. We fix some basis B of K over $\phi(K)$.

3.11 Remark. One has a natural bijection $K^B \longrightarrow K$ given by $(\lambda_b)_{b\in B} \mapsto \sum_{b\in B} \phi(\lambda_b)b$. Let T be the inverse of this map. Denote by T_I the component-wise extension of T to column vectors in K^I (so $T_I : K^I \longrightarrow K^{(I\times B)}$) and by $T_I^i : K^I \longrightarrow K^{I\times B^i}$ for $i \in \mathbb{N}$ the composition $T_{I\times B^{i-1}} \circ \cdots \circ T_I$, so T_I^0 is the identity on K^I and $T_I^{i+1} = T_{I\times B^i} \circ T_I^i$ for all $i \in \mathbb{N}$. If the context is clear, the index I will be dropped in the notation T_I and T_I^i.

3.12 Definition (row and column enlargement of matrices). Let I, J be finite sets.

1. For $M \in K[\Phi]^{I\times J}$, define the **column enlargement** $\mathrm{col}(M) \in (K[\Phi]\Phi)^{I\times(J\times B)}$ of M by $\mathrm{col}(M)(i,(j,b)) := M(i,j)b\Phi$. One can also do column enlargement for a subset J_0 of J: Define $\mathrm{col}_{J_0}(M) := M\restriction_{I\times(J\setminus J_0)} \cup \mathrm{col}(M\restriction_{I\times J_0})$ (regarding the sets $J \setminus J_0$ and $J_0 \times B$ as disjoint); thus, $\mathrm{col}_{J_0}(M) \in K[\Phi]^{I\times J^*}$ where $J^* = (J \setminus J_0) \cup (J_0 \times B)$.

2. For $M \in (K[\Phi]\Phi)^{I\times J}$, define the **row enlargement** $\mathrm{row}(M) \in K[\Phi]^{(I\times B)\times J}$ of M by the equations $M(i,j) = \sum_{b\in B}(b\Phi)\,\mathrm{row}(M)((i,b),j)$. Because B is a basis of K over $\phi(K)$, for every element $f \in K[\Phi]\Phi$ there are unique $f_b \in K[\Phi]$ with $f = \sum_{b\in B}(b\Phi)f_b$. This is easy to see for $f \in K\Phi$ (then the f_b are in K) and the general case follows by right linearity of the relation over $K[\Phi]$.

   For $I_0 \subseteq I$ and $M \in K[\Phi]^{I\times J}$ with $M\restriction_{I_0\times J} \in (K[\Phi]\Phi)^{I_0\times J}$, define (regarding the sets $I \setminus I_0$ and $I_0 \times B$ as disjoint)

   $$\mathrm{row}_{I_0}(M) := M\restriction_{(I\setminus I_0)\times J} \cup \mathrm{row}(M\restriction_{I_0\times J})\,;$$

   thus, $\mathrm{row}_{I_0}(M) \in K[\Phi]^{I^*\times J}$ where $I^* = (I \setminus I_0) \cup (I_0 \times B)$.

   In the following, row and row_{I_0} are viewed as partial functions on $K[\Phi]^{I\times J}$, i.e. saying "$\mathrm{row}_{I_0}(M)$ is defined" means $M\restriction_{I_0\times J} \in (K[\Phi]\Phi)^{I_0\times J}$.

3.13 Remark. Let I, J be finite sets, $M \in K[\Phi]^{I\times J}$.

1. $\mathrm{col}(M) = (\mathrm{col}((M(i,j))))_{(i,j)\in I\times J}$ where $(M(i,j))$ is considered as a 1-by-1-matrix.

2. $\mathrm{row}(M) = (\mathrm{row}((M(i,j))))_{(i,j)\in I\times J}$ where $(M(i,j))$ is considered as a 1-by-1-matrix and the left side is defined if and only if the right hand side is defined.

3. Let $\lambda \in K$ and consider the 1-by-1 matrix $M = (\,\lambda\,)$. Then $\mathrm{row}(\mathrm{col}(M)) \in \mathrm{MAT}_B(K)$. Considering $V = K$ as a vector space over K via the action $\mu \cdot w = \phi(\mu)w$ for $\mu, w \in K$, the matrix $\mathrm{row}(\mathrm{col}(M))$ represents the K-linear map

   $$V \longrightarrow V\,,\; w \mapsto \lambda w$$

   with respect to the basis B of V.

4. Let $J_0 \subseteq J$ and $I_0 \subseteq I$. Then $\mathrm{col}_{J_0}(\mathrm{row}_{I_0}(M)) = \mathrm{row}_{I_0}(\mathrm{col}_{J_0}(M))$ whenever $\mathrm{row}_{I_0}(M)$ is defined.

5. Let L be a finite set and $N \in K[\Phi]^{J \times L}$. Then $\operatorname{col}(MN) = \operatorname{col}(M) \operatorname{row}(\operatorname{col}(N))$.

3.14 Lemma. *Let I, J be finite sets, $M \in K[\Phi]^{I \times J}$. Then there exist $t \in \mathbb{N}$, finite sets $I_1, \ldots, I_t$ and for each $i \in \{1, \ldots, t\}$ a partial operation op_i on $K[\Phi]^{I_{i-1} \times J}$ taking values in $K[\Phi]^{I_i \times J}$, with $I_0 = I$, such that:*

1. *For $i = 1, \ldots, t$, one of the following conditions hold:*

   (a) *$I_i = I_{i-1}$ and op_i is left multiplication by a restricted elementary matrix in $\mathrm{MAT}_{I_i}(K[\Phi])$.*

   (b) *For some $\tilde{I} \subseteq I_{i-1}$, the operator op_i is $\mathrm{row}_{\tilde{I}}$ and $I_i = (I_{i-1} \setminus \tilde{I}) \dot\cup (\tilde{I} \times B)$.*

2. *$\hat{M} := \mathrm{op}_t \circ \cdots \circ \mathrm{op}_1(M)$ is defined and a matrix in $K[\Phi]^{I_t \times J}$ in upper triangular form whose non-zero part is row regular.*

Proof. The proof is similar to that of Lemma 3.9. Again, we do induction on the cardinality of J. Let $M \in K[\Phi]^{I \times J}$ and assume the lemma holds for column sets of smaller cardinality than J. If all entries of M are zero, then M is already in upper triangular form with respect to $I_0 = \emptyset$, and the non-zero part of M is the empty matrix, which is row regular.

Otherwise, consider the natural number $m := \min_{i \in I, j \in J} \operatorname{ldeg} M(i,j)$. Then M is an element of $(K[\Phi]\Phi^m)^{I \times J}$ and we can perform row enlargement on the matrix m times to get a matrix $\tilde{M} \in K[\Phi]^{(I \times B^m) \times J}$ with $\min_{i' \in I \times B^m, j \in J} \operatorname{ldeg} \tilde{M}(i', j) = 0$. Then we proceed as in Lemma 3.9 to find a product E_2 of restricted elementary matrices (in $\mathrm{MAT}_{I \times B^m}(K[\Phi])$) such that $E_2 M$ has a column with exactly one non-zero entry, and this entry has ldeg 0. Now, one can apply the induction hypothesis in a similar way as in the proof of Lemma 3.9. Note that here it is essential to do the induction on the number of columns, since the number of rows of the involved matrices may increase during the procedure. □

The following lemma is essential in the proof of Proposition 7.27.

3.15 Lemma. *Let I, J_x, J_y be finite sets, J_x and J_y being disjoint, and let $J = J_x \cup J_y$. Let $M \in K[\Phi]^{I \times J}$. Then there exist $c, t \in \mathbb{N}$, finite sets $I_1, \ldots, I_t$ and for each $i \in \{1, \ldots, t\}$ a partial operation op_i on $K[\Phi]^{I_{i-1} \times \tilde{J}}$ taking values in $K[\Phi]^{I_i \times \tilde{J}}$, with $I_0 = I$ and $\tilde{J} = (J_x \times B^c) \dot\cup J_y$, such that:*

1. *For $i = 1, \ldots, t$, one of the following conditions holds:*

   (a) *$I_i = I_{i-1}$ and op_i is left multiplication by a restricted elementary matrix in $\mathrm{MAT}_{I_i}(K[\Phi])$;*

   (b) *for some $I^* \subseteq I_{i-1}$, the operator op_i is row_{I^*} and $I_i = (I_{i-1} \setminus I^*) \dot\cup (I^* \times B)$.*

2. *With $\tilde{M} := (\mathrm{col}_{J_x \times B^{c-1}} \circ \cdots \circ \mathrm{col}_{J_x \times B^0})(M) \in K[\Phi]^{I \times \tilde{J}}$, where $J_x \times B^0$ is identified with J_x in the obvious way, the matrix $\hat{M} := (\mathrm{op}_t \circ \cdots \circ \mathrm{op}_1)(\tilde{M}) \in K[\Phi]^{I_t \times \tilde{J}}$ is defined and has the following form: There exists $\hat{I} \subseteq I_t$ such that*

   (a) *$\hat{M}\restriction_{(I_t \setminus \hat{I}) \times J_y}$ is a row regular matrix in $K[\Phi]^{(I_t \setminus \hat{I}) \times J_y}$ in upper triangular form;*

(b) $\hat{M}\restriction_{\hat{I}\times(J_x\times B^c)}$ *is a matrix in* $K[\Phi]^{\hat{I}\times(J_x\times B^c)}$ *in upper triangular form whose non-zero part is row regular;*

(c) $\hat{M}\restriction_{\hat{I}\times J_y} = 0$.

Here is a picture of $\hat{M}$*:*

	$J_x \times B^c$	J_y
$I_t \setminus \hat{I}$	$*$	*upper triangular, row regular*
$\hat{I}$	*upper triangular, non-zero part row regular*	0

Proof. Consider the submatrix $M\restriction_{I\times J_y}$ and find operations op'_i for $i = 1,\dots,t'$ as in the previous lemma to convert this submatrix to upper triangular form whose non-zero part is row regular. Let $I_1,\dots,I_{t'}$ be the corresponding finite sets such that op'_i is a partial operation on $K[\Phi]^{I_{i-1}\times J_y}$ with values in $K[\Phi]^{I_i\times J_y}$. Set $I_0 = I$. Let c be the number of indices $i \in \{1,\dots,t'\}$ such that op'_i is a row enlargement, and let $\tilde{M} := (\mathrm{col}_{J_x\times B^{c-1}} \circ \cdots \circ \mathrm{col}_{J_x\times B^0})(M)$. We now let op_i for $i = 1,\dots,t'$ be the partial operation on $K[\Phi]^{I_{i-1}\times\tilde{J}}$ with values in $K[\Phi]^{I_i\times\tilde{J}}$ defined as follows: if op'_i is left multiplication with the restricted elementary matrix E in $\mathrm{MAT}_{I_{i-1}}(K[\Phi])$, then op_i is also left multiplication with E; if op'_i is row_{I^*} for some $I^* \subseteq I_{i-1}$, then op_i is row_{I^*}. One verifies easily that then $\hat{M}' := (\mathrm{op}_{t'} \circ \cdots \circ \mathrm{op}_1)(\tilde{M})$ is defined and has the following form: There exists $\hat{I}' \subseteq I_{t'}$ such that

1. $\hat{M}'\restriction_{(I_{t'}\setminus\hat{I}')\times J_y}$ is a row regular matrix in $K[\Phi]^{(I_{t'}\setminus\hat{I}')\times J_y}$ in upper triangular form.

2. $\hat{M}'\restriction_{\hat{I}'\times J_y} = 0$.

Here is a picture of $\hat{M}'$:

	$J_x \times B^c$	J_y
$I_{t'} \setminus \hat{I}'$	$*$	upper triangular, row regular
$\hat{I}'$	$*$	0

Now, we apply the previous lemma to the matrix $\hat{M}'\restriction_{\hat{I}' \times (J_x \times B^c)}$ and view the corresponding operations as acting on the whole matrix $\hat{M}'$. In this case, extending row enlargement is not a problem, since $\hat{M}'\restriction_{\hat{I}' \times J_y} = 0$. ☐

3.16 Definition. Let I be a finite (index) set.

1. Let $f = (f_i)_{i \in I} \in K[\Phi]^I$. The **degree** $\deg f$ of f is defined to be $\max_{i \in I} \deg f_i$ where $\max \emptyset := -\infty$. If f has degree less than or equal to $d \in \mathbb{N}$, then f can be expressed as $\sum_{j \le d} v_j \Phi^j$ with $v_j \in K^I$ (where K^I is viewed as embedded in $K[\Phi]^I$). If $\deg f = d$, call v_d the **leading coefficient vector** and $G(f) := v_d \phi^d(K) \subseteq K^I$ the **associated subgroup** of f.

2. Let J be an index set. A tuple $(f_j)_{j \in J}$ over $K[\Phi]^I$ is called **strongly independent**, if all f_j are non-zero and the tuple of additive subgroups $(G(f_j))_{j \in J}$ is independent, i.e. for all $(v_j)_{j \in J} \in \prod_{j \in J} G(f_j)$ one has

$$\sum_{j \in J} v_j = 0 \implies v_j = 0 \text{ for all } j \in J\,.$$

3.17 Remark. It is easy to see (and shown in Section 4.1) that a strongly independent tuple $(f_j)_{j \in J}$ over $K[\Phi]^I$ is independent in the sense of $K[\Phi]^I$ as a right $K[\Phi]$-module. The converse is not true.

3.18 Proposition. *Let I, J be finite sets, $M \in K[\Phi]^{I \times J}$. Then there exists a (possibly empty) product E of restricted elementary matrices in $\mathrm{MAT}_J(K[\Phi])$ such that non-zero column vectors of ME are strongly independent, i.e. setting $f_j := ((ME)(i,j))_{i \in I} \in K[\Phi]^I$ and $J_{\neq 0} = \{\, j \in J \mid f_j \neq 0 \,\}$ the tuple $(f_j)_{j \in J_{\neq 0}}$ is strongly independent.*

Proof. The statement is proved by induction on the well-founded pre-order on $K[\Phi]^{I \times J}$ that is given by comparing the degrees of matching columns, i.e. $M_1 < M_2$ if $\deg(M_1(i,j))_{i \in I} \le \deg(M_2(i,j))_{i \in I}$ for all $j \in J$ and $\deg(M_1(i,j))_{i \in I} < \deg(M_2(i,j))_{i \in I}$ for some $j \in J$. It would also work to do the induction on the sum of the degrees of the non-zero columns of M.

So assume the statement holds for all matrices that are smaller than M in the above sense. Define $f_j := (M(i,j))_{i\in I} \in K[\Phi]^I$ for $j \in J$ and let $J_{\neq 0} := \{ j \in J \mid f_j \neq 0 \}$.

Case a) The tuple of non-zero elements among the f_j is strongly independent, i.e. $(f_j)_{j\in J_{\neq 0}}$ is independent. Then the choice $E := \mathrm{Id}_J$ works to satisfy the statement.

Case b) $(f_j)_{j\in J_{\neq 0}}$ is dependent. Then pick a minimal set $J' \subseteq J_{\neq 0}$ such that $(f_j)_{j\in J'}$ is dependent. For $j \in J'$, let d_j be the degree and v_j the leading coefficient vector of f_j (note that $f_j \neq 0$). Pick $k \in J'$ such that $d_k = \max\{ d_j \mid j \in J' \}$, and set $f = f_k$, $v = v_k$, $d = d_k$. Because $(f_j)_{j\in J'}$ is dependent, there are $\lambda_j \in K$ for $j \in J'$ such that not all λ_j are zero and

$$0 = \sum_{j\in J'} v_j \phi^{d_j}(\lambda_j)\,.$$

By the minimal choice of J', all λ_j are non-zero; in particular $\lambda := \lambda_k$ is non-zero. Then, setting $\mu_j := \lambda_j(\phi^{d-d_j}(\lambda))^{-1}$ for $j \in J'$ and $J'' = J' \setminus \{ k \}$, the following holds:

$$0 = v + \sum_{j\in J''} v_j \phi^{d_j}(\mu_j)\,.$$

Now, let $g_j := \mu_j \Phi^{d-d_j} \in K[\Phi]$. Then $f' := f + \sum_{j\in J''} f_j g_j$ has degree $\leq d = \deg f$, because $\deg g_j = d - d_j = \deg(f) - \deg(f_j)$. The term of degree d of f' is

$$\begin{aligned} & v\Phi^d + \sum_{j\in J''} v_j \Phi^{d_j} \mu_j \Phi^{d-d_j} \\ =& v\Phi^d + \sum_{j\in J''} v_j \phi^{d_j}(\mu_j)\Phi^d \\ =& 0\,. \end{aligned}$$

So performing the appropriate column operations on M yields a matrix $\tilde{M}$ with the same columns as M except the k-column, which has smaller degree than the corresponding column in M. Multiplication with restricted elementary matrices on the right corresponds to performing column operations.

□

3.19 Proposition. *Let I, J be finite sets, $M \in K[\Phi]^{I\times J}$ and $\leq$ be a total order on I. Then there exists a (possibly empty) product E of restricted elementary matrices in $\mathrm{MAT}_J(K[\Phi])$ and pairwise disjoint (possibly empty) subsets $J_i \subseteq J$ for $i \in I$ such that for all $i \in I$:*

1. *$(ME)(i,j) = 0$ for all $j \in J \setminus \bigcup_{l\leq i} J_l$.*
2. *$(ME)(i,j) \neq 0$ for all $j \in J_i$.*
3. *The tuple $((ME)(i,j))_{j\in J_i}$ over $K[\Phi]$ is strongly independent.*

Here is a picture of ME with $I = \{1, \dots, n\}$ in the natural ordering, $J_\infty := J \setminus \bigcup_{i \in I} J_i$, $v_i := ((ME)(i,j))_{j \in J_i}$:

1	v_1	0	0	0	0	0
2	$*$	v_2	0	0	0	0
$\vdots$	$*$	$*$	$\ddots$	0	0	0
$n-1$	$*$	$*$	$*$	v_{n-1}	0	0
n	$*$	$*$	$*$	$*$	v_n	0
	J_1	J_2	$\cdots$	J_{n-1}	J_n	J_∞

Proof. The proof operates by induction on the number of rows (i.e. the cardinality of the set I). Assume $I \neq \emptyset$, $i_1 := \min I$ and that the proposition holds for all matrices with row index set $I_2 := I \setminus \{i_1\}$. Then it suffices to show the following:

Claim. There is a product of restricted elementary matrices $E_1 \in \mathrm{MAT}_J(K[\Phi])$ and a subset $J_1 \subseteq J$ such that

1. $(ME_1)(i_1, j) = 0$ for all $j \in J \setminus J_1$;
2. $(ME_1)(i_1, j) \neq 0$ for all $j \in J_1$;
3. the tuple $((ME_1)(i_1, j))_{j \in J_1}$ of elements in $K[\Phi]$ is strongly independent.

The claim follows by applying the previous proposition to the matrix $M{\restriction}_{\{i_1\} \times J}$.

Using this claim, we apply the induction hypothesis to the matrix $M' := ME_1{\restriction}_{I_2 \times J^*}$ where $J^* := J \setminus J_1$, to obtain a product E_2 of restricted elementary matrices in $\mathrm{MAT}_{J_2}(K[\Phi])$ and pairwise disjoint subsets $J_i \subseteq J_2$ for $i \in I_2$ satisfying the analogue of the claims of the proposition for M'. Then, setting $J_{i_1} := J_1$ and $E = E_1 \hat{E}_2$ where $\hat{E}_2 := \mathrm{Id}_{J_1} \sqcup E_2$, one obtains the data that satisfy the conclusion of the proposition for M. $\square$

Chapter 4

Finitely generated free modules over $K[\Phi]$

4.1 $K[\Phi]^m$ as right $K[\Phi]$-module

Assume K is a field and ϕ is a self-embedding of K such that $[K : \phi(K)]$ is finite. Let I be a finite (index) set and consider $N := K[\Phi]^I$ as a right $K[\Phi]$-module (the action just being right multiplication on the entries of the tuples). For $n \in \mathbb{N}$, let $N_{\leq n} := \{ f \in N \mid \deg f \leq n \}$, and $N_{<n} := \{ f \in N \mid \deg f < n \}$. Also define $N_n := \{ (\mu_i \Phi^n)_{i \in I} \mid \mu_i \in K \text{ for } i \in I \}$.

4.1 Remark.

1. The sets $N_{\leq n}$, $N_{<n}$ and N_n are right K-vector subspaces of the right K-vector space N, where K acts via its inclusion in $K[\Phi]$.

2. We have $N_{\leq n} = N_{<n} \oplus N_n$, $N_n \Phi = N_{n+1}$. Every $f \in N$ can be uniquely written as $\sum_i f_i$ with $f_i \in N_i$ and only finitely many f_i being non-zero. We call f_i the **degree** i **component** of f.

3. Suppose V is a right K-vector space and $n \in N$. We give V a new right K-vector space structure $V_\odot$ via $v \odot \mu := v\phi^n(\mu)$ for $\mu \in K$ and $v \in V$. Then

   $$\dim_K(V_\odot) = \dim_K(V)[K : \phi(K)]^n .$$

   The right K-vector space N_n is isomorphic to $K_\odot^I$ via $(\mu_i \Phi^n)_{i \in I} \mapsto (\mu_i)_{i \in I}$.

4. $\dim_K(N_n) = |I|[K : \phi(K)]^n$, $\dim_K(N_{\leq n}) = |I| \sum_{i \leq n} [K : \phi(K)]^i$.

5. In the rest of the section, N will always be understood as a right $K[\Phi]$-module or with the above defined (right) K-vector space structure, so we will simply talk about $K[\Phi]$-modules or K-vector spaces.

4.2 Remark. Suppose J, K are finite sets, $T \in K[\Phi]^{I \times J}$, $E \in K[\Phi]^{J \times K}$. Let M be the submodule of $N = K[\Phi]^I$ generated by the columns of T and M' be the submodule of N generated by the columns of TE. Then $M' \subseteq M$. If $K = J$ and $E \in \mathrm{GL}_J(K[\Phi])$, then $M = M'$.

4.3 Lemma. *Suppose $(f_j)_{j<n}$ with $f_j \in N$ is strongly independent (see Definition 3.16). Then for all $(g_j)_{j<n} \in K[\Phi]^n$, we have*

$$\deg(\sum_{j<n} f_j g_j) = \max_{j<n}(\deg f_j + \deg g_j)\,.$$

In particular, $(f_j)_{j<n}$ is independent in the right $K[\Phi]$-module N, i.e. for all $(g_j)_{j<n} \in K[\Phi]^n$,

$$\sum_{j<n} f_j g_j = 0 \implies g_j = 0 \textit{ for all } j < n\,.$$

Proof. Clearly, $\deg f_j g_j = \deg f_j + \deg g_j$ for every $j < n$, and so

$$\deg(\sum_{j<n} f_j g_j) \leq \max_{j<n}(\deg f_j + \deg g_j)\,.$$

If all the g_j are zero, the statement is clear. So assume now that at least one g_j is non-zero. Let $d := \max_{j<n} \deg(f_j g_j)$ and note that $0 \leq d$. Let $J := \{\, j < n \mid \deg(f_j g_j) = d \,\}$ and for $j \in J$ let $\mu_j \in K$ be the leading coefficient of g_j and $v_j \in K^I$ be the leading coefficient vector of f_j. The degree d component of $\sum_{j<n} f_j g_j$ is

$$\begin{aligned}\sum_{j\in J} v_j \Phi^{\deg f_j} \mu_j \Phi^{\deg g_j} &= \sum_{j\in J} v_j \phi^{\deg f_j}(\mu_j) \Phi^{\deg f_j + \deg g_j} \\ &= (\sum_{j\in J} v_j \phi^{\deg f_j}(\mu_j)) \Phi^d\,.\end{aligned}$$

We have $v_j \phi^{\deg f_j}(\mu_j) \in G(f_j) \setminus \{\,0\,\}$, since $\mu_j \neq 0$. Because the f_j are strongly independent, $\sum_{j\in J} v_j \phi^{\deg f_j}(\mu_j) \neq 0$ holds, so $\deg(\sum_{j<n} f_j g_j) \geq d$. □

4.4 Remark. Not every independent tuple in N is strongly independent: If $t \in K \setminus \phi(K)$, then for $f_1 = \Phi^2 + t\Phi$ and $f_2 = \Phi^2$ the tuple (f_1, f_2) is not strongly independent, but independent (in $N = K[\Phi]$) because $(f_1 + f_2(-1), f_2) = (t\Phi, \Phi^2)$ is strongly independent, so independent.

4.5 Lemma. *Let V be a K-subspace of $N_{\leq n}$. Let M be the $K[\Phi]$-submodule generated by V in N. Then*

$$V\Phi \cap N_{\leq n} \subseteq V \iff M \cap N_{\leq n} = V\,.$$

Proof. The direction from right to left is clear, since $V\Phi \subseteq M$. To show the other implication, assume $V\Phi \cap N_{\leq n}$. Because V is a K-subspace of N, it suffices to show that

$$(V\Phi^0 + \cdots + V\Phi^l) \cap N_{\leq n} \subseteq V$$

for all $l \in \mathbb{N}$. This will be proved by induction on l. We have $V\Phi^0 = V$, so the case $l = 0$ is clear.

One has

$$(V\Phi^0 + \cdots + V\Phi^{l+1}) \cap N_{\leq n} = (V + (V\Phi^0 + \cdots + V\Phi^l)\Phi) \cap N_{\leq n}$$
$$\subseteq V + ((V\Phi^0 + \cdots + V\Phi^l)\Phi \cap N_{\leq n}),$$

since $V \subseteq N_{\leq n}$. So it suffices to show $(V\Phi^0 + \cdots + V\Phi^l)\Phi \cap N_{\leq n} \subseteq V$. Let

$$x \in (V\Phi^0 + \cdots + V\Phi^l)\Phi \cap N_{\leq n}.$$

Then $x = x'\Phi$ for some $x' \in V\Phi^0 + \cdots + V\Phi^l$. Because $x \in N_{\leq n}$, the element x' lies in $N_{<n} \subseteq N_{\leq n}$. Now by the induction assumption $x' \in V$, and therefore $x \in V\Phi \cap N_{\leq n}$, which by assumption is a subset of V. □

4.6 Remark. The map $f : N_{\leq n} \longrightarrow N_{\leq n+1}$, $v \mapsto v\Phi$ is in general not K-linear, but it is K-linear, where $N_{\leq n}$ is equipped with the K-vector space structure given by $v \odot \mu := v\phi(\mu)$ for $v \in N_{\leq n}$ and $\mu \in K$: $f(v \odot \mu) = f(v\phi(\mu)) = v\phi(\mu)\Phi = v\Phi\mu = f(v)\mu$.

Suppose V is a K-subspace of $N_{\leq n}$ (with respect to the original K-space structure of $N_{\leq n}$). Then it is also a K-subspace with respect to the structure given by $\odot$, and denoted by $V_\odot$ as a vector space with this structure. If T is a basis of V (in the sense of the original K) and B is a basis of K over $\phi(K)$, then $\{ v\mu \mid v \in T, \mu \in B \}$ is a basis of $V_\odot$. In particular, the dimension of $V_\odot$ is $\dim_K(V)[K : \phi(K)]$.

4.7 Remark. Let $S \subseteq N$ be a finite set and M the $K[\Phi]$-submodule generated by S. One can effectively find a K-basis of $M \cap N_{\leq n}$ (modulo performing field operations in K, checking equality for elements in K and applying ϕ).

Here is an explanation: Note that $N_{\leq i}$ is an effective (right) K-vector space for every $i \in \mathbb{N}$ (see Remark 4.1). We may assume that $S \subseteq N_{\leq n}$. Otherwise, just choose $n' > n$ so that $S \subseteq N_{\leq n'}$, and after computing a basis of $M \cap N_{\leq n'}$, compute a basis of the intersection with $N_{\leq n}$, which is just linear algebra over K.

By Remark 4.6, the map $f : N_{\leq n} \longrightarrow N_{\leq n+1}$, $v \mapsto v\Phi$ is K-linear, if one equips $N_{\leq n}$ with the structure given by $\odot$. Given a basis T of V with respect to the original structure, one can obtain one with respect to the $\odot$ structure and then compute a basis of $f(V)$ using K-linear algebra.

If one defines V_0 as the K-span of S and then inductively $V_{i+1} = (V_i + V_i\Phi) \cap N_{\leq n}$, one gets an increasing sequence of K-subspaces of $N_{\leq n}$: $V_i \subseteq V_{i+1} \subseteq N_{\leq n}$ for all i. So for some $i_0 \leq \dim_K(N_{\leq n})$, one has $V_{i_0} = V_j$ for all $j \geq i_0$. In particular, $V_{i_0}\Phi \cap N_{\leq n} \subseteq V_{i_0}$, so $M \cap N_{\leq n} = V_{i_0}$ by Lemma 4.5.

So to determine a basis of $M \cap N_{\leq n}$, one only has to successively determine bases of the V_i for $i = 0, \ldots, \dim_K(N_{\leq n})$.

4.8 Proposition. *Suppose M is a finitely generated submodule of N and $n_1, n_2 \in \mathbb{N}$ are such that $n_2 \geq n_1 \geq \deg g$ for all g in some set of generators of M. Then*

$$\dim_K((M \cap N_{\leq n_1})/(M \cap N_{<n_1}))[K : \phi(K)]^{n_2 - n_1} = \dim_K((M \cap N_{\leq n_2})/(M \cap N_{<n_2}));$$

in other words, if π_i denotes the canonical map $N_{\leq i} \longrightarrow N_{\leq i}/N_{<i}$ for $i = n_1, n_2$, then

$$\dim_K(\pi_{n_1}(M \cap N_{\leq n_1}))[K : \phi(K)]^{n_2 - n_1} = \dim_K(\pi_{n_2}(M \cap N_{\leq n_2}))\,.$$

4.9 Definition. For M as in the previous proposition, define $\dim_\infty M$ as the value of

$$\frac{\dim_K((M \cap N_{\leq n})/(M \cap N_{<n}))}{[K : \phi(K)]^n}$$

for $n \geq n_1$.

Proof of Proposition 4.8. After proving the case $n_2 = n_1 + 1$, the general statement follows by induction. So let $n \in \mathbb{N}$ be greater than or equal to the maximum of the degrees of some set of generators of M.

Claim. $M \cap N_{\leq n+1} = (M \cap N_{\leq n}) + (M \cap N_{\leq n})\Phi$.

The inclusion $\supseteq$ is obvious. For the other inclusion, let $W = M \cap N_{\leq n}$ and $V = W + W\Phi$. Then the claim says that $M \cap N_{\leq n+1} = V$. Since M is generated by elements in $N_{\leq n}$, it is generated by W, hence by $V \supseteq W$. Also $V \subseteq N_{\leq n+1}$, so by Lemma 4.5 it suffices to show that $V\Phi \cap N_{\leq n+1} \subseteq V$. Let $v \in V$ be such that $v\Phi \in N_{\leq n+1}$. Then $v \in N_{\leq n}$. Also $v \in M$, since $W \subseteq M$, so $v \in W$. Therefore $v\Phi \in W\Phi \subseteq V$, and the claim is proved.

The map

$$f : N_{\leq n} \longrightarrow N_{\leq n+1}\,,\ v \mapsto v\Phi$$

induces a bijection $\bar{f}$ between $N_{\leq n}/N_{<n}$ and $N_{\leq n+1}/N_{<n+1}$, and the claim shows that

$$\bar{f}(\pi_n(M \cap N_{\leq n})) = \pi_{n+1}(M \cap N_{\leq n+1})\,.$$

A direct consequence of this last equality and Remark 4.6 is that

$$\dim_K(\pi_{n+1}(M \cap N_{\leq n+1})) = [K : \phi(K)]\dim_K(\pi_n(M \cap N_{\leq n}))\,.$$

□

4.10 Definition. Assume B is a basis of K over $\phi(K)$.

1. For $b = (b_k)_{k<n} \in B^n$, the **basis polynomial** $q_b^B \in K[\Phi]$ **with respect to** B **and** b is $\prod_{k<n} b_k \Phi$.

2. For $f \in N$ and $b = (b_k)_{k<n} \in B^n$, let

$$f_b := f q_b^B \in N$$

and call the tuple $(f_b)_{b \in B^n}$ the n**-enlargement of** f **with respect to** B. Call the K-subspace of N generated by the f_b the n**-enlargement space** of f (this is independent of

B).

3. For a tuple $(f_j)_{j\in J}$ where J is some finite (index) set and $f_j \in N \setminus \{0\}$ for $j \in J$ and $n \in \mathbb{N}$, define
$$J_{B,n} := \left\{ (j,b) \mid j \in J, \deg f_j \le n, b = (b_k)_{k<n-\deg f_j} \in B^{n-\deg f_j} \right\}.$$
For $(j,b) \in J_{B,n}$ with $b = (b_k)_{k<n-\deg f_j}$, let $f_{(j,b)}$ be the entry at index b of the $(n - \deg f_j)$-enlargement of f_j with respect to B, i.e.
$$f_{(j,b)} := f_j \prod_{k<n-\deg f_j} b_k\Phi\,,$$
and call the tuple $(f_{(j,b)})_{(j,b)\in J_{B,n}}$ the **common n-enlargement of $(f_j)_{j\in J}$ with respect to B**.

4. For $b = (b_k)_{k<n} \in B^n$, the **basis element $\omega_b^B \in K$ with respect to B and b** is $\prod_{k<n} \phi^k(b_k)$. Note that $(\omega_b^B)_{b\in B^n}$ is a basis of K over $\phi^n(K)$, so for every $\mu \in K$, there exist unique scalars $\lambda_b^B(\mu) \in K$ such that
$$\mu = \sum_{b\in B^n} \phi^n(\lambda_b^B(\mu))\omega_b^B\,.$$

4.11 Remark. Assume B is a basis of K over $\phi(K)$.

1. Viewing $f \in N$ as a matrix M in $K[\Phi]^{I\times\{0\}}$, the n-enlargement of f with respect to B is just obtained by applying column enlargements to M, i.e. the n-enlargement of f with respect to B corresponds to the matrix
$$\mathrm{col}_{\{0\}\times B^{n-1}}(\ldots \mathrm{col}_{\{0\}\times B}(\mathrm{col}_{\{0\}}(M))\ldots)\,.$$
A similar statement holds for the common enlargement of a tuple $(f_j)_{j\in J}$ viewing it as an element of $K[\Phi]^{I\times J}$.

2. $q_b^B = \omega_b^B\Phi^n$ for $b \in B^n$ and $(q_b^B)_{b\in B^n}$ is a (right) K-basis of $K[\Phi]_n = \{ \mu\Phi^n \mid \mu \in K \}$, since $\mu\Phi^n = \sum_{b\in B^n} q_b^B \lambda_b^B(\mu)$ for all $\mu \in K$.

3. Let $(f_i)_{i\in I}$ be the tuple of elements in N indexed by I corresponding to the identity matrix in $K[\Phi]^{I\times I}$, i.e. for $i, j \in I$ let $f_i(j) = 1$ if $j = i$ and $f_i(j) = 0$ otherwise.

   Then the common n-enlargement of $(f_i)_{i\in I}$ with respect to B is a K-basis of N_n.

4. All entries in a common n-enlargement with respect to B have degree equal to n.

5. Let J be a finite index set, $(f_j)_{j\in J} \in N^J$ and $n_j \in \mathbb{N}$ for $j \in J$. For $j \in J$, let $(f_{j,b})_{b\in B^{n_j}}$ be the n_j-enlargement of f_j with respect to B. Then $(f_j)_{j\in J}$ is strongly independent if and only if $(f_{j,b})_{j\in J, b\in B^{n_j}}$ is strongly independent.

This follows from $G(f) = \oplus_{b\in B^n} G(f_b)$ for any $f \in N \setminus \{0\}$ and its corresponding n-enlargement $(f_b)_{b\in B^n}$ with respect to B.

4.12 Proposition. *Suppose M is a finitely generated submodule of N. There exist a finite set J and a strongly independent tuple $(f_j)_{j\in J} \in N^J$ such that $\{ f_j \mid j \in J \}$ generates M; in particular M is free. Suppose that $(f_j)_{j\in J}$ is such a strongly independent generating tuple for M. Then:*

1. *If S is a generating set of M, then $\max_{g\in S} \deg g \geq \max_{j\in J} \deg f_j$ (with $\max \emptyset := -\infty$).*
2. *Let B be a basis of K over $\phi(K)$, $d \in \mathbb{N}$ and $\pi_d : N_{\leq d} \longrightarrow N_{\leq d}/N_{<d}$ the canonical map.*
   - *(a) Let $(f_{(j,b)})_{(j,b)\in J_{B,d}}$ be the common d-enlargement of $(f_j)_{j\in J}$ with respect to B. Then $(\pi_d(f_{(j,b)}))_{(j,b)\in J_{B,d}}$ is a K-basis of $\pi_d(M \cap N_{\leq d})$.*
   - *(b) Put*

     $$J^* := \left\{ (j, b) \mid j \in J, \deg f_j \leq d, b \in B^i \text{ , where } i \in \mathbb{N} \text{ satisfies } 0 \leq i \leq d - \deg f_j \right\}$$

     and let $f_{j,b}$ be the entry at index b of the i-enlargement of f_j with respect to B, for $(j, b) \in J^*$ and $b \in B^i$. Then $(f_{j,b})_{(j,b)\in J*}$ is a K-basis of $M \cap N_{\leq d}$.
3. *$\dim_\infty M = \sum_{j\in J} [K : \phi(K)]^{-\deg f_j}$.*

Proof. The existence of a strongly independent generating tuple $(f_j)_{j<n} \in N^J$ for M follows from Remark 4.2 and Proposition 3.18. Suppose that $(f_j)_{j\in J}$ is such a strongly independent generating tuple for M.

By Lemma 4.3, every element $g \in M$ is contained in the submodule generated by the f_j with $\deg f_j \leq \deg g$. In particular, if a set of generators of M would contain only elements that have strictly lower degree than f_{j_0} for some $j_0 \in J$, then this set of generators is contained in the submodule generated by $\{ f_j \mid j \in J \setminus \{ j_0 \} \}$, so a proper subset of $\{ f_j \mid j \in J \}$ generates M. This contradicts the fact that the f_j are independent by Lemma 4.3 and establishes statement 1.

To prove statement 2, fix a basis B of K over $\phi(K)$ and $d \in \mathbb{N}$, and note that $M \cap N_{\leq d}$ is generated by $\{ f_j \mid j \in J, \deg f_j \leq d \}$. So for the purpose of proving statement 2, we can assume that $\deg f_j \leq d$ for all $j \in J$.

By Remark 4.11, the common d-enlargement $(f_{(j,b)})_{(j,b)\in J_{B,d}}$ of $(f_j)_{j\in J}$ with respect to B is strongly independent and, since all the polynomials in the enlargement have the same degree d, the tuple $(\pi_d(f_{(j,b)}))_{(j,b)\in J_{B,d}}$ is K-independent. To prove that this tuple is a K-basis of $\pi_d(M)$, it suffices to show that the K-vector space that the $f_{(j,b)}$ (with $(j, b) \in J_{B,d}$) together with $N_{<d}$ generate contains $M \cap N_{\leq d}$. So let $h \in M \cap N_{\leq d}$. Then there are $g_j \in K[\Phi]$ for $j \in J$ such that $h = \sum_{j\in J} f_j g_j$. By Lemma 4.3, $\deg g_j \leq d - \deg f_j$ for all $j \in J$. For $j \in J$, let $\mu_j \Phi^{d-\deg f_j}$ be the component of degree $d - \deg f_j$ of g_j (with $\mu_j \in K$). Using Remark 4.11, one has $\mu_j \Phi^{d-\deg f_j} =$

$\sum_{b\in B^{d-\deg f_j}} q_b^B \lambda_b^B(\mu_j)$, so the degree d component of $f_j g_j$ is equal to the degree d component of

$$f_j \mu_j \Phi^{d-\deg f_j} = \sum_{b\in B^{d-\deg f_j}} (f_j q_b^B)\lambda_b^B(\mu_j)$$

and the elements $(f_j q_b^B)$ constitute the common d-enlargement of $(f_j)_{j\in J}$ with respect to B. Therefore, h lies in the K-span of this enlargement together with $N_{<d}$. Part 2a is now proved, and part 2b follows easily by induction on d using part 2a.

For statement 3, let $d \in \mathbb{N}$ such that $d \geq \max_{j\in J} \deg f_j$, pick some basis B of K over $\phi(K)$ and apply part 2a to conclude that

$$\dim_K((M\cap N_{\leq d})/(M\cap N_{<d})) = |J_{B,d}| = \sum_{j\in J}[K:\phi(K)]^{d-\deg f_j}$$

and therefore

$$\dim_\infty M = \frac{\dim_K((M\cap N_{\leq d})/(M\cap N_{<d}))}{[K:\phi(K)]^d} = \sum_{j\in J}[K:\phi(K)]^{-\deg f_j}\,.$$

□

4.2 $K[\Phi]^m$ as left $K[\Phi]$-module

Assume K is a field, ϕ a self-embedding of K.

Let J be a finite (index) set and consider $N := K[\Phi]^J$ as a left $K[\Phi]$-module (the action just being left multiplication on the entries of the tuples). Also I is assumed to be a finite set in this section.

4.13 Remark. Let $M \in K[\Phi]^{I\times J}$. Then for every matrix E in $\mathrm{MAT}_I(K[\Phi])$ the submodule of N generated by the rows of EM is contained in the submodule of N generated by the rows of M. In particular, if E is invertible, then the submodule generated by the rows of M and the submodule generated by the rows of EM are the same.

4.14 Lemma. *Every submodule M of N has an independent generating set of cardinality less than or equal to $|J|$.*

Proof. This follows from the previous remark and Lemma 3.9, using the fact that N is noetherian.

□

4.15 Definition. Let M be a submodule of N. Call M **separable**, if for all finite $B \subseteq K$ that are independent over $\phi(K)$ and for all $(x_b)_{b\in B} \in N^B$ we have

$$\sum_{b\in B}(b\Phi)x_b \in M \implies (x_b \in M \text{ for all } b\in B)\,.$$

4.16 Remark. The submodules N and $\{0\}$ of N are separable.

4.17 Remark/Definition. Every subset S of N is contained in a smallest separable submodule M of N, which is called the **separable submodule generated by** S.

4.18 Remark. Suppose $[K : \phi(K)]$ is finite and B is a basis of K over $\phi(K)$.

1. Let M be a submodule of N and α be an automorphism of the left $K[\Phi]$-module N. Then M is separable if and only if $\alpha(M)$ is separable.

2. A submodule M of N is separable if and only if for all $(x_b)_{b\in B} \in N^B$ one has

$$\sum_{b\in B}(b\Phi)x_b \in M \implies (x_b \in M \text{ for all } b \in B)\,.$$

3. Suppose $M \in K[\Phi]^{I\times J}$, $I_0 \subseteq I$ and $M' = \mathrm{row}_{I_0}(M)$ (the row enlargement of M with respect to the basis B) exists. Then the separable submodule of N generated by the rows of M is the same as the separable submodule of N generated by the rows of M'.

4.19 Definition.

1. Let I be an index set. A tuple $(f_i)_{i\in I}$ with all $f_i \in N$ is called **left strongly independent**, if the tuple $(\overline{f_i})_{i\in I}$ is independent over K in K^J where $\overline{}$ is the ring homomorphism $K[\Phi] \longrightarrow K$ with $\overline{\lambda} = \lambda$ for all $\lambda \in K$ and $\overline{\Phi} = 0$.

2. Let M be a submodule of N, and M' be the separable submodule it generates. Define $\dim_0 M := \dim_K \overline{M'}$.

4.20 Remark. Let $f = (f_i)_{i\in I} \in N^I$.

1. If the tuple f is left strongly independent, then it is independent in the left module N: Suppose $g_i \in K[\Phi]$ for $i \in I$ and $\sum_{i\in I} g_i f_i = 0$. Then $\sum_{i\in I} \overline{g_i}\,\overline{f_i} = 0$, so for $i \in I$ we have $g_i \in K[\Phi]\Phi$ and we can write $g_i = \sum_{b\in B} b\Phi g_{i,b}$ with $g_{i,b} \in K[\Phi]$ for $b \in B$. Because $\{0\}$ is separable, we obtain $\sum_{i\in I} g_{i,b} f_i = 0$ for $b \in B$. If m is a bound on the degrees of the g_i, then repeating this procedure m times yields $g_i = 0$ for all $i \in I$.

2. The tuple f is left strongly independent if and only if the matrix $((f_i)(j))_{i\in I, j\in J} \in K[\Phi]^{I\times J}$ is row regular.

3. Let M be a submodule of N and α be an automorphism of the left $K[\Phi]$-module N. Then $\dim_0 M = \dim_0 \alpha(M)$.

For the rest of the section, assume that $[K : \phi(K)]$ is finite and B is a basis of K over $\phi(K)$.

4.21 Lemma.

1. *If* $f = (f_i)_{i\in I} \in N^I$ *is left strongly independent, then the submodule* M *generated by* $\{ f_i \mid i \in I \}$ *is separable, and thus* $\dim_0 M = |I|$.

2. *Suppose* M *is a separable submodule of* N. *Then there exists a left strongly independent tuple over* M *whose entries generate* M.

Proof. For the first part, let $F = (f_i)_{i\in I} \in N^I$ be left strongly independent, and let M be the submodule generated by the f_i, $i \in I$. Let $x_b \in N$ for $b \in B$ and assume

$$(*) \qquad \sum_{b\in B} (b\Phi)x_b \in M\,.$$

By Remark 4.18, it suffices to show that $x_b \in M$ for all $b \in B$.

Regard F as an element of $K[\Phi]^{I\times J}$ and $X = (x_b)_{b\in B}$ as an element of $K[\Phi]^{B\times J}$. Regard $c = (b\Phi)_{b\in B}$ as a row vector (with column index set B). Then (*) can be expressed as the existence of $d = (d_i)_{i\in I} \in K[\Phi]^I$ (regarded as a row vector with column index set I) such that $cX = dF$. For such d, we have

$$\overline{dF} = \overline{c}\overline{X} = 0\overline{X} = 0$$

and because $\overline{F}$ has full row rank this implies $\overline{d} = 0$. Therefore, there exist $G \in K[\Phi]^{B\times I}$ such that $d_i = \sum_{b\in B} b\Phi G(b, i)$ for all $i \in I$. This yields $X = GF$, so every row x_b of X lies in M.

This argument can also be phrased in terms of a dual notion of row and column enlargement.

For the second part, choose a matrix $H \in K[\Phi]^{I\times J}$ for some index set I such that the rows of H generate M. By Lemma 3.14, there exists a matrix $\tilde{H} \in K[\Phi]^{\tilde{I}\times J}$ that is obtained from H via the operations of multiplying by restricted elementary matrices from the left and row enlargement for subsets of the rows such that $\hat{H}$ is in upper triangular form whose non-zero part is row regular. By Remark 4.13 and Remark 4.18, the mentioned operations do not change the separable submodule generated by the rows of the involved matrices, so the separable module generated by the rows of $\hat{H}$ is M. The non-zero rows of $\hat{H}$ form a left strongly independent tuple and by the first part of the lemma the submodule generated by this tuple is separable. Therefore, the non-zero rows of $\hat{H}$ form a generating set for M. □

4.22 Remark. 1. The proof of the second part of the previous lemma also indicates a procedure to compute a left strongly independent generating tuple for the separable module generated by some elements of N: Suppose the elements are the rows of a matrix $H \in K[\Phi]^{I\times J}$ for some index set I. If $\tilde{H} \in K[\Phi]^{\tilde{I}\times J}$ is obtained from H according to Lemma 3.14, then the non-zero rows of $\tilde{H}$ form a left strongly independent generating tuple for the separable submodule generated by the rows of H.

2. One can compute $\dim_0 M$ of a submodule M of N in the following way: Compute a left strongly independent generating tuple $f = (f_i)_{i\in I} \in N^I$ for the separable submodule generated by M. Then $\dim_0 M = |I|$.

4.23 Remark/Definition. If we regard the elements of $N = K[\Phi]^J$ as matrices with a single row and column index set J, then column enlargement with respect to the basis B yields a homomorphism col $:K[\Phi]^J \longrightarrow K[\Phi]^{(J\times B)}$ of left $K[\Phi]$-modules. So, if M is a submodule of N, then $\mathrm{col}(M)$ is a submodule of $K[\Phi]^{(J\times B)}$. The module $\mathrm{col}(M)$ is called the **column-enlargement of M with respect to B.**

4.24 Lemma. *Let M be a submodule of N. Then* $\dim_0 \mathrm{col}(M) = |B| \dim_0 M$.

Proof. Let M' be the separable submodule generated by M and $\mathrm{col}(M)'$ be the separable submodule generated by $\mathrm{col}(M)$. Note that $\mathrm{col}(M') \subseteq \mathrm{col}(M)'$, since the inverse image of a separable submodule of $K[\Phi]^{(J\times B)}$ under the homomorphism col is a separable submodule of N. So the separable submodule generated by $\mathrm{col}(M')$ is $\mathrm{col}(M)'$.

Pick a finite set I with $|I| = \dim_0 M$ and a row regular matrix $H \in K[\Phi]^{I\times J}$ whose rows generate the submodule M'. By Remark 4.18, part 3, the rows of $\mathrm{row}(\mathrm{col}(H))$ and $\mathrm{col}(M')$ generate the same separable submodule, which is $\mathrm{col}(M)'$.

Note that $\overline{\mathrm{row}(\mathrm{col}(H))} = \mathrm{row}(\mathrm{col}(\overline{H}))$. Remark 3.13 shows that

$$\mathrm{rank}_K \mathrm{row}(\mathrm{col}(\overline{H})) = |B| \, \mathrm{rank}_K \overline{H} \,,$$

so $\mathrm{rank}_K \overline{\mathrm{row}(\mathrm{col}(H))} = |B||I|$, and therefore $\mathrm{row}(\mathrm{col}(H))$ is row regular. By Lemma 4.21, the submodule generated by the rows of $\mathrm{row}(\mathrm{col}(H))$ is separable, so the rows of $\mathrm{row}(\mathrm{col}(H))$ generate $\mathrm{col}(M)'$, and we get

$$\dim_0 \mathrm{col}(M) = |B||I| = |B| \dim_0 M \,.$$

□

4.25 Remark. There exist separable submodules M_1 and M_2 of $K[\Phi]$ such that $M_1 \subsetneqq M_2$ and $\dim_0 M_1 = \dim_0 M_2$: Let M_1 be the left submodule of $K[\Phi]$ generated by $1+\Phi$ and $M_2 = K[\Phi]$. Then both M_1 and M_2 are separable with $\dim_0 M_1 = \dim_0 M_2 = 1$, and $M_1 \subsetneqq M_2$.

Let $\mathfrak{m} = K[\Phi]\Phi$ and $K[\Phi]_{\mathfrak{m}}$ be the ring of left fractions of $K[\Phi]$ with respect to the multiplicative set $S = K[\Phi] \setminus \mathfrak{m}$ (see Chapter 2 , Remark 2.9 and Lemma 2.10). We consider $K[\Phi]$ as a subring of $K[\Phi]_{\mathfrak{m}}$ and $N_{\mathfrak{m}} := K[\Phi]_{\mathfrak{m}}^J$ as a left $K[\Phi]_{\mathfrak{m}}$-module.

4.26 Remark/Definition. Let M be a submodule of N. Then

$$M_{\mathfrak{m}} := \left\{ \, (g^{-1} f_j)_{j\in J} \;\middle|\; g \in S, (f_j)_{j\in J} \in M \, \right\}$$

is a submodule of $N_{\mathfrak{m}}$. This submodule $M_{\mathfrak{m}}$ is the submodule generated by the subset M in $N_{\mathfrak{m}}$.

4.27 Remark. By the universal property of the ring of fractions for $K[\Phi]_{\mathfrak{m}}$, there exists a unique ring homomorphism $\overline{} :K[\Phi]_{\mathfrak{m}} \longrightarrow K$ extending $\overline{} :K[\Phi] \longrightarrow K$. For every submodule M of N, we have $\overline{M} = \overline{M_{\mathfrak{m}}}$.

4.28 Lemma. *Let M_1, M_2 be separable submodules of N such that $M_1 \subseteq M_2$. Then* $\dim_0 M_1 = \dim_0 M_2$ *if and only if* $(M_1)_{\mathfrak{m}} = (M_2)_{\mathfrak{m}}$.

Proof. Suppose $(M_1)_{\mathfrak{m}} = (M_2)_{\mathfrak{m}}$. Then by the previous remark $\overline{M_1} = \overline{M_2}$ and since M_1 and M_2 are separable, this implies $\dim_0 M_1 = \dim_0 M_2$.

For the other direction, assume $\dim_0 M_1 = \dim_0 M_2$. Since M_1 is separable, there exists $J_0 \subseteq J$ and a row regular matrix $M \in K[\Phi]^{J_0 \times J}$ that is in upper triangular form with respect to J_0 and $\iota = \mathrm{id}_{J_0}$ such that the rows of M generate M_1. Given $f \in M_2$ and using that the diagonal elements of M are invertible in $K[\Phi]_{\mathfrak{m}}$, one can find an element $g \in (M_1)_{\mathfrak{m}}$ such that for $f' = f - g$ we have $f'_j = 0$ for $j \in J_0$. Write $f' = h^{-1} f''$ with $h \in S$ and $f'' \in M_2$. It suffices to show that $f'' = 0$. For a contradiction, assume that this is not the case.

Claim. Suppose $h \in M_2 \setminus \{0\}$ and $h_j = 0$ for all $j \in J_0$. Then there exists $h' \in M_2 \setminus \{0\}$ with $h'_j = 0$ for all $j \in J_0$ and $h'_j \in S$ for some $j \in J \setminus J_0$.

The claim follows by induction on $\min_{j \in J} \operatorname{ldeg} h_j$ using the the separability of M_2 .

Applying the claim for $h = f''$, we obtain $\dim_K(\overline{M_2}) > |J_0| = \dim_K(\overline{M_1})$, contradicting $\dim_0 M_1 = \dim_0 M_2$. □

Chapter 5

K as left $K[\Phi]$-module and its definable sets

Assume K is a field and ϕ is a self-embedding of K.

5.1 Definition. We expand the additive group of K to a left $K[\Phi]$-module by requiring that for all $\lambda, \mu \in K$ one has

$$(\lambda\Phi^0) \cdot \mu := \lambda\mu$$

and

$$\Phi \cdot \mu := \phi(\mu)\,.$$

5.2 Remark. The map $K \longrightarrow K[\Phi]/K[\Phi](\Phi - 1)\,,\ \lambda \mapsto \lambda + K[\Phi](\Phi - 1)$ is an isomorphism of left $K[\Phi]$-modules. For $f = \sum_{i \le n} \lambda_i \Phi^i \in K[\Phi]$ with all $\lambda_i \in K$, and $\mu \in K$, one has

$$f \cdot \mu = \sum_{i \le n} \lambda_i \phi^i(\mu)\,.$$

Let I, J be finite index sets.

5.3 Definition.

1. For $M \in K[\Phi]^{I \times J}$ and $\lambda \in K^J$, define $M \cdot \lambda \in K^I$ by

   $$(M \cdot \lambda)(i) = \sum_{j \in J} M(i,j) \cdot \lambda(j)$$

   for $i \in I$. Similarly, we define $f \cdot \lambda := \sum_{j \in J} f_j \cdot \lambda_j \in K$ for $f \in K[\Phi]^J$, $\lambda \in K^J$ and $f \cdot \lambda := (f_i \cdot \lambda)_{i \in I} \in K^I$ for $f \in K[\Phi]^I$ and $\lambda \in K$.

2. For $S \subseteq K[\Phi]^I$ and $\Lambda \subseteq K$, set $S \cdot \Lambda := \{\, f \cdot \lambda \mid f \in S, \lambda \in \Lambda \,\}$. Similarly, define $S \cdot \lambda := S \cdot \{\, \lambda \,\}$ and $f \cdot \Lambda := \{\, f \,\} \cdot \Lambda$.

3. For a subset S of $K[\Phi]^J$, define the **annihilator** of S in K as

   $$\mathrm{Ann}(S) := \left\{\, \lambda \in K^J \;\middle|\; f \cdot \lambda = 0 \text{ for all } f \in S \,\right\}\,.$$

5.4 Remark.

1. Suppose $S \subseteq K[\Phi]^I$. If M is the right $K[\Phi]$-submodule generated by S in $K[\Phi]^I$, then $\sum_{f\in S} f\cdot K = M\cdot K = M\cdot 1 \subseteq K^I$. If M is the right K-subspace generated by S in $K[\Phi]^I$, then $\sum_{f\in S} f\cdot K = M\cdot K = M\cdot 1 \subseteq K^I$.

2. Suppose $S \subseteq K[\Phi]^J$ and M is the separable left $K[\Phi]$-submodule generated by S in $K[\Phi]^J$. Then $\mathrm{Ann}(S) = \mathrm{Ann}(M)$.

3. For $M \in K[\Phi]^{H\times I}$ (with finite H), $N \in K[\Phi]^{I\times J}$ and $\lambda \in K^J$, one has $(MN)\cdot\lambda = M\cdot(N\cdot\lambda)$.

We will consider the left $K[\Phi]$-module K as a structure in the following signature: $\sigma_{K[\Phi]} = (+, 0, -, (f\cdot)_{f\in K[\Phi]})$, where $+$ is interpreted as the addition in K, 0 as the zero of K, $-$ as the additive inverse of K (considered as a unary function), and $f\cdot$ as the unary function given by module multiplication with the ϕ-polynomial f, for $f \in K[\Phi]$.

5.5 Remark. If in addition to $+$, 0 and $-$ we only add functions symbols $f\cdot$ for a set of f that generates the ring $K[\Phi]$ (for example for $f \in K \cup \{\, \Phi \,\}$), then the structure in this reduced signature has the same definable sets over the empty set and hence over an arbitrary subset of K.

If we specify some additive subgroups of K, we can expand the signature by adding unary predicates for them. Given a set $\mathcal{P}$ of symbols for unary predicates, we consider the signature

$$\sigma_{K[\Phi],\mathcal{P}} = \sigma_{K[\Phi]} \sqcup (P)_{P\in\mathcal{P}}\,,$$

and K becomes a structure in this signature, if we interpret each $P \in \mathcal{P}$ as an additive subgroup of K. For convenience, we assume that $\mathcal{P}$ always contains a predicate V_∞ for the zero subgroup.

5.6 Example. Let (K, v, Γ) be a valued field and put $\mathcal{P} := \{\, V_\gamma \mid \gamma \in \Gamma \cup \{\, \infty \,\} \,\}$. Interpret V_γ in K as the additive subgroup $\{\, \lambda \in K \mid v(\lambda) \geq \gamma \,\}$. Note that in the signature $\sigma_{K[\Phi],\mathcal{P}}$ the structure K has the same definable sets as in the reduced signature $\sigma_{K[\Phi],\{\, V_0 \,\}}$ (over the empty set, and hence over an arbitrary subset of K), because $\lambda V_0 = V_{v(\lambda)}$ for $\lambda \in K$.

5.7 Convention. We fix here some notation for (first order) formulas. In referring to $\tau(X)$ **as a formula**, we mean that τ is a formula, X is a finite set of variables and the free variables of τ are in X; formally we regard $\tau(X)$ as a pair (τ, X) to specify the indexing set for the solutions. Similarly, in referring to $\tau(X, Y)$ as a formula, we mean that τ is a formula, X and Y are finite disjoint sets of variables and the free variables of τ are in $X \cup Y$. For a formula $\tau(X, Y)$, we let $\exists Y\, \tau(X, Y)$ denote a formula $\tau'(X) = \exists y\, \tau$ where y is a tuple of distinct variables enumerating Y. Similarly, we use $\forall Y\, \tau(X, Y)$.

For a formula $\tau(X)$ of signature σ, $\mathcal{M}$ a σ-structure, and $a \in \mathcal{M}^X$, the statement "$\mathcal{M} \models \tau(a)$" has the obvious meaning, and $\tau(\mathcal{M}) := \{\, a \in \mathcal{M}^X \mid \mathcal{M} \models \tau(a) \,\}$.

Now we recall some results on the model theory of modules with distinguished subgroups.

5.8 Definition. Let $T_{K[\Phi],\mathcal{P}}$ consist of the following sentences in the language determined by $\sigma_{K[\Phi],\mathcal{P}}$ where $f, g \in K[\Phi]$ and $P \in \mathcal{P}$:

1. $\forall x, y, z\, \big((x + y) + z = x + (y + z) \wedge x + y = y + x \wedge x + 0 = x \wedge x + (-x) = 0\big)$,
2. $\forall x, y\, \big(f \cdot (x + y) = (f \cdot x) + (f \cdot y)\big)$,
3. $\forall x\, \big((f + g) \cdot x = (f \cdot x) + (g \cdot x)\big)$,
4. $\forall x\, \big((fg) \cdot x = f \cdot (g \cdot x)\big)$,
5. $\forall x\, \big(1 \cdot x = x\big)$,
6. $P(0) \wedge \forall x, y\, \big((P(x) \wedge P(y)) \to (P(x + y) \wedge P(-x))\big)$ and
7. $\forall x\, \big(V_\infty(x) \to x = 0\big)$.

These sentences express that a model of $T_{K[\Phi],\mathcal{P}}$ is a module over $K[\Phi]$ with distinguished subgroups given by the predicates $P \in \mathcal{P}$ and that V_∞ is the trivial subgroup.

The structure K as defined above clearly satisfies the sentences in $T_{K[\Phi],\mathcal{P}}$. It also satisfies sentences that are not logical consequences of these, but these basic module axioms already ensure that in each model the 0-definable sets are boolean combinations of sets defined by pp-formulas as defined below.

5.9 Remark. Suppose X is a finite set of variables, I some finite (index) set, $M \in K[\Phi]^{I \times X}$ and $P = (P_i)_{i \in I} \in \mathcal{P}^I$. Then $M \cdot X \in P$ stands for the conjunction of atomic formulas

$$\bigwedge_{i \in I} P_i\Big(\sum_{x \in X} M(i, x) \cdot x\Big)$$

with distinguished set of variables X. In a model $\mathcal{M}$ of $T_{K[\Phi],\mathcal{P}}$, every such formula $\tau(X)$ defines an additive subgroup of the product group $\mathcal{M}^X$ (but not a submodule in general). Also every conjunction of atomic formulas is equivalent modulo $T_{K[\Phi],\mathcal{P}}$ to a formula of this form (note that one of the axioms says that V_∞ is the trivial subgroup, so one does not need equations).

5.10 Definition. A **positive primitive formula** (short: pp-formula) is a formula in any language of the form $\exists Y\, \tau(X, Y)$ where $\tau(X, Y)$ is a conjunction of atomic formulas. If I is a finite set, X and Y are finite disjoint sets of variables, $M \in K[\Phi]^{I \times (X \cup Y)}$ and $P \in \mathcal{P}^I$, then $\exists Y\, (M \cdot (X \cup Y) \in P)$ is a pp-formula in the signature $\sigma_{K[\Phi],\mathcal{P}}$. A pp-formula of this form is called **special**.

5.11 Remark. In a model $\mathcal{M}$ of $T_{K[\Phi],\mathcal{P}}$, every pp-formula $\tau(X)$ defines an additive subgroup of the product group $\mathcal{M}^X$. Every such formula $\tau(X)$ is equivalent modulo $T_{K[\Phi],\mathcal{P}}$ to a special pp-formula $\exists Y\, (M \cdot (X \cup Y) \in P)$.

The following is taken from Hodges' book [Ho] with small changes in the formulation:

Let L be a a language whose signature includes symbols $\cdot$ (binary function symbol), ι (unary function symbol) and 1 (constant symbol). Let the L-structure $\mathcal{M}$ be **group-like**, that is, $\cdot^{\mathcal{M}}$, $\iota^{\mathcal{M}}$ and $1^{\mathcal{M}}$ are the multiplication, inversion and identity of a group with the same underlying set as $\mathcal{M}$. A pp-formula $\tau(X)$ is called **basic in** $\mathcal{M}$, if $\tau(\mathcal{M})$ is a subgroup of the product group $\mathcal{M}^X$. For pp-formulas $\alpha(X)$, $\beta(X)$ in L and $m \geq 1$, let $\mathrm{Inv}_{\alpha(X),\beta(X),m}$ be an L-sentence such that for each group-like L-structure $\mathcal{M}$ such that $\alpha(X)$ and $\beta(X)$ are basic in $\mathcal{M}$,

$$\mathcal{M} \models \mathrm{Inv}_{\alpha(X),\beta(X),m} \iff |\alpha(\mathcal{M})/(\alpha(\mathcal{M}) \cap \beta(\mathcal{M}))| \leq m\,.$$

It is easy to construct such a sentence $\mathrm{Inv}_{\alpha(X),\beta(X),m}$ from $\alpha(X)$, $\beta(X)$, m. Such a sentence $\mathrm{Inv}_{\alpha(X),\beta(X),m}$ is called an invariant sentence.

5.12 Fact ([Ho], theorem A.1.1). *Let X be a finite set of variables, y a single variable not in X, $\Psi(X,y)$ a finite set of pp-formulas in L and $\psi(X,y)$ a boolean combination of formulas in Ψ.*

Then there exists a finite set Θ of pp-formulas in L and a formula $\theta(X)$ that is a boolean combination of formulas in Θ and invariant sentences $\mathrm{Inv}_{\alpha(z),\beta(z),m}$ with $\alpha(z), \beta(z) \in \Theta$ and z a single variable, such that the following holds:

If $\mathcal{M}$ is a group-like L-structure in which all formulas of Ψ are basic, then all formulas in Θ are basic in $\mathcal{M}$ and $\exists y\, \psi(X,y)$ is equivalent to $\theta(X)$ in $\mathcal{M}$.

A consequence of this is the following quantifier elimination theorem obtained by Baur and Monk (see [Ba], [Mo]):

5.13 Theorem ([Ho], corollary A.1.2). *Every formula $\tau(X)$ in the language of signature $\sigma_{K[\Phi],\mathcal{P}}$ is equivalent modulo $T_{K[\Phi],\mathcal{P}}$ to a boolean combination of pp-formulas and invariant sentences. In particular, every complete theory in the signature $\sigma_{K[\Phi],\mathcal{P}}$ extending $T_{K[\Phi],\mathcal{P}}$ is axiomatized by $T_{K[\Phi],\mathcal{P}}$ and a set of invariant sentences.*

So to determine the complete theory of K as a $\sigma_{K[\Phi],\mathcal{P}}$-structure, it suffices to determine for every two special pp-formulas $\alpha(z)$ and $\beta(z)$ of signature $\sigma_{K[\Phi],\mathcal{P}}$, the index $|\alpha(K)/(\alpha(K)\cap\beta(K))| \in \mathbb{N} \cup \{\,\infty\,\}$.

Chapter 6

Valued settings and Estimates

Let (K, v, Γ) be a valued field with valuation v and value group Γ. We assume that the valuation v maps $K^\times$ onto Γ.

6.1 Definition. A **valued vector space over** (K, v, Γ) consists of a K-vector space W, a totally ordered set Δ, an action $+ : \Gamma \times \Delta \longrightarrow \Delta$ of the group Γ on Δ and a surjective map $v : W \setminus \{ 0 \} \longrightarrow \Delta$ such that the following properties hold, where $\Delta_\infty := \Delta \dot\cup \{ \infty_\Delta \}$ with the ordering on Δ extended to a total order on Δ_∞ by $\infty_\Delta > \delta$ for all $\delta \in \Delta$ and where v on $W \setminus \{ 0 \}$ is extended to W via $v(0) := \infty_\Delta$: For all $a, b \in W \setminus \{ 0 \}$ and $\lambda, \mu \in K^\times$,

1. $v(a + b) \geq \min \{ v(a), v(b) \}$,
2. $v(\lambda a) = v(\lambda) + v(a)$, and
3. the action of Γ on Δ preserves the order in the following way: $v(\lambda) \leq v(\mu) \implies v(\lambda a) \leq v(\mu a)$, and $v(a) \leq v(b) \implies v(\lambda a) \leq v(\lambda b)$.

From now on in this chapter, (W, Δ, v) is a valued vector space over (K, v, Γ). For $S \subseteq \Gamma \cup \{ \infty_\Gamma \}$, we let $\min S$ denote the minimum of S if it exists, with the convention that $\min \emptyset = \infty_\Gamma$. A similar convention holds for $S \subseteq \Delta \cup \{ \infty_\Delta \}$.

6.2 Remark.

1. It is convenient to extend the action of Γ on Δ to that of the ordered semigroup $\Gamma_\infty := \Gamma \dot\cup \{ \infty_\Gamma \}$ on Δ_∞ via $\infty_\Gamma + \delta = \gamma + \infty_\Delta = \infty_\Delta$ for all $\gamma \in \Gamma_\infty$ and $\delta \in \Delta_\infty$. Then the three properties above hold for all $a, b \in W$ and $\lambda, \mu \in K$.
2. Let $|_K$ and $|_W$ be the binary relations on K and W respectively, given by the equivalences

   $$\lambda |_K \mu \iff v(\lambda) \leq v(\mu)$$

   and

   $$a |_W b \iff v(a) \leq v(b) \,.$$

   Then for all $a, b, c \in W$ and $\lambda, \mu \in K$:

   (a) if $a|_W b$ and $b|_W c$, then $a|_W c$;

(b) $a|_W b$ or $b|_W a$;

(c) if $a|_W b$ and $a|_W c$, then $a|_W b + c$;

(d) if $a|_W b$, then $\lambda a|_W \lambda b$;

(e) if $\lambda|_K \mu$, then $\lambda a|_W \mu a$.

These properties (a)-(e) are just a translation of the definition of *valued vector space* in terms of the divisibility relations $|_K$ and $|_W$.

3. The structure (K, v, Γ) is a valued vector space over itself, with the action being the addition on Γ.

4. The K-vector space W^m (with operations defined componentwise) becomes a valued vector space over (K, v, Γ) by setting $v((w_i)_{i<m}) := \min_{i<m} v(w_i)$ for $(w_i)_{i<m} \in W^m$.

5. The structure (W, Δ, v) is also naturally a valued vector space over any valued subfield of (K, v, Γ).

6.3 Definition. Let $a = (a_i)_{i\in I}$ be a finite tuple over W. Then a is called **weakly valuation independent over** K (in the valued vector space (W, Δ, v)), if there exists $\delta \in \Delta$ such that for all $(\lambda_i)_{i\in I} \in K^n$,

$$v(\sum_{i\in I} \lambda_i a_i) \leq v(\lambda_j) + \delta \text{ for } j \in I\,.$$

We call a **weakly valuation independent** over a subfield K_0, if it is weakly valuation independent with W being regarded as a valued vector space over K_0 with its induced valuation. We call a **valuation independent over** K, if $a_i \neq 0$ for $i \in I$ and for all $(\lambda_i)_{i\in I} \in K^n$,

$$v(\sum_{i\in I} \lambda_i a_i) = \min_{i\in I} v(\lambda_i) + v(a_i)\,.$$

The tuple a is said to be a **weak valuation basis of** W **over** K, if a is weakly valuation independent over K and a basis of W over K. Similarly, a is said to be a **valuation basis of** W **over** K, if a is valuation independent over K and a basis of W over K.

6.4 Remark.

1. Let $a = (a_i)_{i\in I} \in W^I$ be a finite tuple. If a is weakly valuation independent in the valued vector space (W, Δ, v) over K, then it is linearly independent over K.

2. If $a = (a_i)_{i\in I}$ is a valuation basis of K over a subfield K_0, then a is weakly valuation independent over K_0 (with K a valued vector space over K_0 in the natural way).

6.5 Definition/Convention.

1. Recall that by our notational conventions $W^{n\times m}$ is the set of matrices over W with row index set $\{0, \ldots, n-1\}$ and column index set $\{0, \ldots, m-1\}$.

2. For a matrix $M \in W^{n\times m}$, define $v(M) := \min\{\, v(M(i,j)) \mid i < n, j < m \,\}$.

3. Tuples $x = (x_i)_{i<n} \in W^n$ are identified with elements of $W^{1\times n}$ (row vectors) or $W^{n\times 1}$ (column vectors) as appropriate.

4. For matrices $M_1 \in K^{n\times m}$ and $M_2 \in W^{m\times r}$, define $M_1M_2 \in W^{n\times r}$ by

$$(M_1M_2)(i,j) = \sum_{k<m} M_1(i,k)M_2(k,j)\,.$$

5. The above conventions and definitions are later also applied, if (W, v, Δ) is a valued field extension of (K, v, Γ).

6.6 Remark.

1. Let $M \in K^{m\times n}$ and consider $a = (a_i)_{i<n} \in W^n$ as a column vector. Then $v(Ma) \geq v(M) + v(a)$. If $M \in \mathrm{GL}_n(K)$, then $(-v(M^{-1})) + v(a) \geq v(Ma) \geq v(M) + v(a)$. Let $\lambda = (\lambda_i)_{i<m}$ and consider λ as a row vector. Then $v(\lambda M) \geq v(\lambda) + v(M)$. If $M \in \mathrm{GL}_m(K)$, then $v(\lambda) - v(M^{-1}) \geq v(\lambda M) \geq v(\lambda) + v(M)$.

2. Let $a = (a_i)_{i<n} \in W^n$ and regard it as a column vector. Then a is weakly valuation independent over K if and only if there exists $\delta \in \Delta$ such that for all $\lambda = (\lambda_i)_{i<n} \in K^n$, which are regarded as row vectors, $v(\lambda a) \leq v(\lambda) + \delta$.

6.7 Lemma. *Let $M \in \mathrm{GL}_n(K)$ and assume $a = (a_i)_{i<n} \in W^n$ is weakly valuation independent over K. Then Ma is weakly valuation independent over K, where a is regarded as a column vector.*

Proof. Pick $\delta \in \Delta$ as in the definition of weak valuation independence for a. Let $\lambda = (\lambda_i)_{i<n} \in K^n$ and regard it as a row vector. Then $\lambda(Ma) = (\lambda M)a$ and by valuation independence of a one has $v((\lambda M)a) \leq v(\lambda M) + \delta$. By the previous remark, $v(\lambda M) \leq v(\lambda) + (-v(M^{-1}))$, so $v(\lambda(Ma)) \leq v(\lambda) + (-v(M^{-1}) + \delta)$. ☐

6.8 Corollary. *If W has a weak valuation basis over K, then every finite tuple over W that is K-linearly independent is weakly valuation independent over K.*

6.9 Lemma. *Consider the valued vector space W^m with $v((a_i)_{i<m}) = \min_{i<m} v(a_i)$. Suppose that the tuple $(b_j)_{j<n}$ over W is weakly valuation independent over K. Let $H := m \times n$ and consider it as an index set for tuples. For $h = (h_1, h_2) \in H$, define $c_h = (\delta_{i,h_1} b_{h_2})_{i<m}$, where $\delta_{i_1,i_2} = 1$ for $i_1 = i_2$ and $\delta_{i_1,i_2} = 0$ otherwise. Then $c = (c_h)_{h\in H}$ is a tuple indexed by H over W^m. It is weakly valuation independent over K in the valued vector space W^m.*

Proof. Pick $\delta \in \Delta$ to witness that $(b_j)_{j<n}$ is weakly valuation independent. Let $\mu = (\mu_h)_{h\in H} \in K^H$. Claim: $v(\mu c) \leq v(\mu) + \delta$.

Let $\pi_i : W^m \longrightarrow W$ denote the canonical projection for $i < m$. We have

$$\begin{aligned}
v(\mu c) &= v(\sum_{h \in H} \mu_h c_h) \\
&= \min_{i<m} v(\pi_i(\sum_{h \in H} \mu_h c_h)) \\
&= \min_{i<m} v(\sum_{h \in H} \mu_h \pi_i(c_h)) \\
&= \min_{i<m} v(\sum_{(h_1,h_2) \in H} \mu_{(h_1,h_2)} \delta_{i,h_1} b_{h_2}) \\
&= \min_{i<m} v(\sum_{j<n} \mu_{(i,j)} b_j) \\
&\leq \min_{i<m} (\min_{j<n} v(\mu_{(i,j)}) + \delta) \\
&= (\min_{(h_1,h_2) \in H} v(\mu_{(h_1,h_2)})) + \delta \\
&= v(\mu) + \delta
\end{aligned}$$

□

6.10 Corollary. *If W is finite dimensional and has a weakly valuation independent basis over K, then every finite tuple over W^m that is K-linearly independent is weakly valuation independent over K in the valued vector space W^m.*

Proof. Apply the lemma to a weakly valuation independent basis of W. Then the tuple c is a basis of W^m and it is weakly valuation independent over K, so by Corollary 6.8, every linearly independent tuple over W^m is weakly valuation independent. □

6.11 Lemma. *Suppose $a = (a_i)_{i<n} \in W^n$ is weakly valuation independent in W over K, K_0 is a subfield of K and $\lambda = (\lambda_j)_{j<m} \in K^m$ is weakly valuation independent in K over K_0. Then for $H = n \times m$ and $b_{(i,j)} = \lambda_j a_i$ for $(i,j) \in H$, the tuple $(b_{(i,j)})_{(i,j)\in H}$ is weakly valuation independent over K_0 in the valued vector space W.*

Proof. Let $(\mu_{(i,j)})_{(i,j)\in H} \in K_0^H$. Then

$$\begin{aligned}
v(\sum_{(i,j)\in H} \mu_{(i,j)} b_{(i,j)}) &= v(\sum_{i<n} (\sum_{j<m} \mu_{(i,j)} \lambda_j) a_i) \\
&\leq \min_{i<n} v(\sum_{j<m} \mu_{(i,j)} \lambda_j) + \delta
\end{aligned}$$

for some $\delta \in \Delta$ independent of $\mu_{(i,j)}$, because a is weakly valuation independent over K. Because λ is weakly valuation independent over K_0, we can find $\gamma \in \Gamma$ independent of $(\mu_{(i,j)})_{(i,j)\in H}$ such that

$$\begin{aligned}\min_{i<n} v(\sum_{j<m} \mu_{(i,j)}\lambda_j) + \delta &\leq \min_{i<n}(\min_{j<m} v(\mu_{(i,j)}) + \gamma) + \delta \\ &= \min_{(i,j)\in H} v(\mu_{(i,j)}) + (\gamma + \delta)\,.\end{aligned}$$

□

In the following, there are some considerations on ordered abelian groups with a specific kind of self-embedding. The purpose is to prepare the setting for the situation, where K is equipped with a specific kind of self-embedding. So assume that $(\Gamma, +, 0, \leq)$ is a an ordered abelian group and ϕ an embedding of this ordered group into itself.

6.12 Definition. A **modulus of growth g for ϕ** is a function $g :\Gamma \longrightarrow \Gamma$ such that for all $\epsilon \in \Gamma$ and all $\gamma \geq g(\epsilon)$,

$$\phi(\gamma) - \gamma \geq \epsilon\,.$$

A **modulus of size s for ϕ** is a function $s :\Gamma \longrightarrow \Gamma$ such that for all $\epsilon \in \Gamma$ and all $\gamma \geq s(\epsilon)$,

$$\phi(\gamma) \geq \epsilon\,.$$

6.13 Remark.

1. If g is a modulus of growth for ϕ, then $s(\epsilon) := \max\{\,\epsilon, g(0)\,\}$ is a modulus of size for ϕ.
2. If g is a modulus of growth for ϕ, then any function $g' :\Gamma \longrightarrow \Gamma$ such that $g'(\gamma) \geq g(\gamma)$ for all γ is also a modulus of growth for ϕ.
3. Assume that there is a rational number $C > 1$ such that $\phi(\epsilon) \geq C\epsilon$ holds for all $\epsilon \in \Gamma$ with $\epsilon \geq 0$. Then the function m defined by $m(\epsilon) := (C-1)^{-1}\epsilon$ for $\epsilon \geq 0$ and $m(\epsilon) = 0$ for $\epsilon < 0$ is a modulus of growth for ϕ, provided Γ is divisible. If Γ is not divisible, picking a natural number $D \geq (C-1)^{-1}$ and setting $m(\epsilon) := D\epsilon$ for $\epsilon \geq 0$ and $m(\epsilon) = 0$ for $\epsilon < 0$, yields a modulus of growth for ϕ.
4. If g_i is a modulus of growth for a self-embedding ϕ_i of Γ for $i = 1, 2$, then $g(\epsilon) := \max\{\,g_2(0), g_1(\epsilon)\,\}$ is a modulus of growth for $\phi_1 \circ \phi_2$. In particular, if g is a modulus of growth for ϕ and $i > 0$ is a natural number, then $g'(\epsilon) := \max\{\,g(0), g(\epsilon)\,\}$ is a modulus of growth for ϕ^i.
5. ϕ is also a self-embedding of $(\Gamma, +, 0, \geq)$, where the order is reversed, and if g is a modulus of growth for ϕ in the structure $(\Gamma, +, 0, \leq)$, then the function defined by $g'(\epsilon) := -g(-\epsilon)$ is a modulus of growth for ϕ in $(\Gamma, +, 0, \geq)$.

Assume below that g_i is a modulus of growth for ϕ^i for $i > 0$ and that s_i is a modulus of size for ϕ^i for $i \geq 0$.

6.14 Lemma. *Given natural numbers* $0 \leq i_0 < i_1$ *and* $\mu_0, \mu_1 \in \Gamma$*, there exist* $\gamma_-, \gamma_+, \delta_-, \delta_+ \in \Gamma$ *such that for all* $\gamma \in \Gamma$*,*

$$(\implies_+) \qquad \gamma \geq \gamma_+ \implies \phi^{i_0}(\gamma) + \mu_0 \leq \phi^{i_1}(\gamma) + \mu_1 \,,$$
$$(\implies_-) \qquad \gamma \leq \gamma_- \implies \phi^{i_0}(\gamma) + \mu_0 \geq \phi^{i_1}(\gamma) + \mu_1 \,,$$

$$(\max) \qquad \phi^{i_0}(\gamma) + \mu_0 \leq \max\left\{\, \phi^{i_1}(\gamma) + \mu_1, \delta_+ \,\right\} \,,$$
$$(\min) \qquad \phi^{i_0}(\gamma) + \mu_0 \geq \min\left\{\, \phi^{i_1}(\gamma) + \mu_1, \delta_- \,\right\} \,.$$

The same also holds for ($\implies_+$) *and* ($\implies_-$) *simultaneously replacing* $\geq$ *by* $>$ *and* $\leq$ *by* $<$*.*

Proof. It suffices to find γ_+, δ_+ and prove ($\implies_+$) and (max). The other two statements ($\implies_-$) and (min) are just the dual of ($\implies_+$) and (max) in the sense of Remark 6.13, part 5.

Set $\gamma_+ = g_{i_1 - i_0}(s_{i_0}(\mu_0 - \mu_1))$. Then for $\gamma \in \Gamma$ with $\gamma \geq \gamma_+$, we have

$$\begin{aligned} \phi^{i_1}(\gamma) - \phi^{i_0}(\gamma) &= \phi^{i_0}(\phi^{i_1 - i_0}(\gamma) - \gamma) \\ &\geq \phi^{i_0}(s_{i_0}(\mu_0 - \mu_1)) \\ &\geq \mu_0 - \mu_1 \,, \end{aligned}$$

so $\phi^{i_0}(\gamma) + \mu_0 \leq \phi^{i_1}(\gamma) + \mu_1$. To obtain the statement with strict inequalities, choose some $\epsilon > 0$ if Γ is non-trivial (otherwise the statement is trivially true) and apply the statement with weak inequalities where μ_0 is replaced by $\mu_0 + \epsilon$.

The choice of $\delta_+ = \phi^{i_0}(\gamma_+) + \mu_0$ obviously works to satisfy (max), because ϕ^{i_0} is monotone. $\square$

6.15 Corollary. *Given* $j \in \mathbb{N}$*, there exists* $\gamma_-, \gamma_+ \in \Gamma$ *such that for all* i_0, i_1 *with* $i_0 \leq i_1 \leq j$ *and all* $\gamma \in \Gamma$*, one has*

$$\phi^{i_0}(\gamma) \leq \max\left\{\, \phi^{i_1}(\gamma), \gamma_+ \,\right\} \,,$$
$$\phi^{i_0}(\gamma) \geq \min\left\{\, \phi^{i_1}(\gamma), \gamma_- \,\right\} \,.$$

6.16 Lemma. *Let* $i_0, i_1 \in \mathbb{N}$ *with* $i_0 < i_1$ *and* $\delta_0, \delta_1, \gamma_0 \in \Gamma$*. Then the set*

$$\left\{\, \gamma \in \Gamma \mid \phi^{i_1}(\gamma) + \delta_1 \geq \min\left\{\, \phi^{i_0}(\gamma) + \delta_0, \gamma_0 \,\right\} \right\}$$

is bounded below.

Proof. By Lemma 6.14, there exist a γ_- such that for all $\gamma < \gamma_-$, one has $\phi^{i_0}(\gamma) + \delta_0 > \phi^{i_1}(\gamma) + \delta_1$ and $\gamma > \phi^{i_1}(\gamma) + \delta_1$.

Let $\gamma \in \Gamma$ and $\phi^{i_1}(\gamma) + \delta_1 \geq \min\left\{ \phi^{i_0}(\gamma) + \delta_0, \gamma_0 \right\}$. If $\gamma < \gamma_-$, then $\phi^{i_1}(\gamma) + \delta_1 \geq \gamma_0$, so $\gamma > \gamma_0$. □

6.17 Remark. Assume that ϕ is a self-embedding of K (only in the sense of pure fields) and that there exists a rational number $C_\phi > 1$ such that for $\lambda \in K$ with $v(\lambda) \geq 0$, the inequality $v(\phi(\lambda)) \geq C_\phi v(\lambda)$ holds. One can interpret the multiplication of an element by a rational number in the (up to isomorphism over Γ) unique divisible hull of Γ, or simply clear denominators in the defining inequality.

For $\lambda, \mu \in K$, if $v(\lambda) \leq v(\mu)$, then $v(\phi(\lambda)) \leq v(\phi(\mu))$. For $\lambda = 0$, this is clear and otherwise this follows from $v(\phi(\frac{\mu}{\lambda})) \geq C_\phi v(\frac{\mu}{\lambda}) \geq 0$. Therefore ϕ induces an embedding (of ordered abelian groups) of the value group into itself, which is also denoted by ϕ, and this map has a modulus of growth by Remark 6.13, part 3.

The map ϕ is actually a self-embedding of (K, v, Γ).

6.18 Assumption. More generally, in the rest of the chapter, ϕ is a self-embedding of (K, v, Γ). Such a self-embedding consist of a self-embedding of the field K and a self-embedding of the value group Γ, which are both denoted by ϕ, such that $v(\phi(\lambda)) = \phi(v(\lambda))$ for all $\lambda \in K^\times$. We assume that $\phi : \Gamma \longrightarrow \Gamma$ has a modulus of growth g, and that Γ is not trivial.

6.19 Lemma. *Let $f = \sum_{i \leq d} \mu_i \Phi^i \in K[\Phi]$, where $d \in \mathbb{N}$ and $\mu_i \in K$ for $i = 0, \ldots, d$.*

1. *Setting $\delta := \min_{i \leq d} v(\mu_i)$, one has*

$$v(f \cdot \lambda) \geq \min_{i \leq d}(\phi^i(v(\lambda)) + v(\mu_i)) \geq \delta + \min_{i \leq d} \phi^i(v(\lambda))$$

   for all $\lambda \in K$.

2. *There exists $\gamma_b \in \Gamma$ such that*

$$v(f \cdot \lambda) \geq \min\left\{ \phi^d(v(\lambda)) + \delta, \gamma_b \right\}$$

   for all $\lambda \in K$.

3. *Suppose $\gamma \in \Gamma$ and $\delta_\gamma = \min_{i \leq d}(v(\mu_i) + \phi^i(\gamma))$. Then for all $\lambda \in K$ with $v(\lambda) \geq \gamma$, the inequality $v(f \cdot \lambda) \geq \delta_\gamma$ holds, hence $\{ v(f \cdot \lambda) \mid v(\lambda) \geq \gamma \}$ is bounded below.*

Proof. Let $\lambda \in K$. Then

$$\begin{aligned} v(f\cdot\lambda) &= v(\sum_{i\le d}\mu_i\phi^i(\lambda)) \\ &\ge \min_{i\le d}(v(\mu_i)+v(\phi^i(\lambda))) \\ &= \min_{i\le d}(v(\mu_i)+\phi^i(v(\lambda))) \\ &\ge \min_{i\le d}(\delta+\phi^i(v(\lambda))) \\ &= \delta+\min_{i\le d}\phi^i(v(\lambda)) \end{aligned}$$

and the parts 1 and 3 follow. For part 2, apply Corollary 6.15 to obtain $\gamma_- \in \Gamma$ such that $\phi^i(\gamma) \ge \min\{\phi^d(\gamma), \gamma_-\}$ for $i \le d$. Then $v(f\cdot\lambda) \ge \delta + \min_{i\le d}\phi^i(v(\lambda)) \ge \delta + \min\{\phi^d(v(\lambda)), \gamma_-\} = \min\{\phi^d(v(\lambda))+\delta, \gamma_-+\delta\}$ so one can choose γ_b as $\gamma_- + \delta$. □

6.20 Proposition. *Let $f = \sum_{d_0\le i\le d_1}\mu_i\Phi^i \in K[\Phi]$, where $\mu_i \in K$, $\operatorname{ldeg} f = d_0$, $\deg f = d_1$. Then there exists $\gamma_-, \gamma_+ \in \Gamma$ such that for all $\lambda \in K$:*

(l) $$v(\lambda) > \gamma_+ \implies \phi^i(v(\lambda)) + v(\mu_i) > v(f\cdot\lambda) = \phi^{d_0}(v(\lambda)) + v(\mu_{d_0})$$

for $d_0 < i \le d_1$ and

(h) $$v(\lambda) < \gamma_- \implies v(f\cdot\lambda) = \phi^{d_1}(v(\lambda)) + v(\mu_{d_1}) < \phi^i(v(\lambda)) + v(\mu_i)$$

for $0 \le i < d_1$.

In particular, the first statement implies that the function $f\cdot : K \longrightarrow K, \lambda \mapsto f\cdot\lambda$ is continuous with respect to the valuation topology on K.

Proof. One has $v(\mu_i\Phi^i)\cdot\lambda) = v(\mu_i) + \phi^i(v(\lambda))$ for $d_0 \le i \le d_1$, so

$$\begin{aligned} v(f\cdot\lambda) &= v((\sum_{d_0\le i\le d_1}\mu_i\Phi^i)\cdot\lambda) \\ &\ge \min_{d_0\le i\le d_1}(v(\mu_i)+\phi^i(v(\lambda))) \end{aligned} \tag{*}$$

and equality holds, if the minimum is attained for a single i.

Applying Lemma 6.14, there are $\gamma_{+,i}$ for $d_0 < i \le d_1$ such that $\gamma > \gamma_{+,i}$ implies

$$\phi^{d_0}(\gamma) + v(\mu_{d_0}) < \phi^i(\gamma) + v(\mu_i)\,.$$

Now assume $d_0 < d_1$ and set $\gamma_+ = \max_{d_0<i\le d_1}\gamma_{+,i}$. Then for $d_0 < i \le d_1$ and $\gamma > \gamma_+$, one has

$$\phi^{d_0}(\gamma) + v(\mu_{d_0}) < \phi^i(\gamma) + v(\mu_i)\,,$$

so in the inequality (*) actually equality holds for $v(\lambda) > \gamma_+$ and the statement (l) is established.

The statement (h) is established similarly. □

6.21 Proposition. *Let* $f = (f_i)_{i<m} \in K[\Phi]^m$ *and* $d \in \mathbb{N}$*, such that* $\deg f_i \leq d$ *for* $i < m$*. Then there exists* $\delta, \gamma_-, \gamma \in \Gamma$ *such that for all* $\lambda \in K$*:*

$$\text{(min)} \qquad v(f \cdot \lambda) \geq \min\left\{ \phi^d(v(\lambda)) + \delta, \gamma \right\}$$

and

$$(*) \qquad v(\lambda) < \gamma_- \implies v(f \cdot \lambda) \geq \phi^d(v(\lambda)) + \delta$$

If one of the f_i *has degree equal to* d*, then one can choose* $\delta, \gamma_- \in \Gamma$ *such that in (*) equality holds.*

Proof. By Lemma 6.19, for $i < m$, there exist $\delta_i, \gamma_i \in \Gamma$ such that

$$v(f_i \cdot \lambda) \geq \min\left\{ \phi^d((v(\lambda)) + \delta_i, \gamma_i \right\}$$

for all $\lambda \in K$. So

$$\begin{aligned} v(f \cdot \lambda) &= \min_{i<m} v(f_i \cdot \lambda) \\ &\geq \min_{i<m} \min\left\{ \phi^d((v(\lambda)) + \delta_i, \gamma_i \right\} \\ &\geq \min\left\{ \phi^d(v(\lambda)) + \delta, \gamma \right\} \end{aligned}$$

for $\delta := \min_{i<m} \delta_i$ and $\gamma := \min_{i<m} \gamma_i$, and (min) is proved.

By deleting the f_i that are zero (which don't influence the value of $v(f \cdot \lambda)$), we may assume that all f_i are non-zero. For each $i < m$, there exists by Proposition 6.20 γ_i, δ_i such that for $\lambda \in K$ with $v(\lambda) < \gamma_i$, the equality $v(f_i \cdot \lambda) = \phi^{d_i}(v(\lambda)) + \delta_i$ holds where $d_i = \deg f_i$. Let d' be the maximum of the d_i and δ be the minimum of the δ_i with $d_i = d'$.

For $i < m$ such that $d_i < d'$, apply Lemma 6.14 to find γ'_i with the property that for all $\gamma \leq \gamma'_i$,

$$\phi^{d_i}(\gamma) + \delta_i \geq \phi^{d'}(\gamma) + \delta$$

holds, and set

$$\gamma_- = \min\left\{ \gamma_i \mid i < m \right\} \cup \left\{ \gamma'_i \,\middle|\, i < m, d_i < d' \right\} .$$

Now let $\lambda \in K$ with $v(\lambda) < \gamma$ and $i < m$. Then $v(f_i \cdot \lambda) = \phi^{d_i}(v(\lambda)) + \delta_i$ and if $d_i = d'$, this is equal to $\phi^{d'}(v(\lambda)) + \delta_i \geq \phi^{d'}(v(\lambda)) + \delta$ and equality holds for some such i. If $d_i < d'$, then $\phi^{d_i}(v(\lambda)) + \delta_i \geq \phi^{d'}(v(\lambda)) + \delta$.

This shows that

$$\begin{aligned} v(f \cdot \lambda) &= \min_{i<m} v(f_i \cdot \lambda) \\ &= \phi^{d'}(v(\lambda)) + \delta \end{aligned}$$

□

6.22 Proposition. *Let $m, d \in \mathbb{N}$ and $f_i \in K[\Phi]^m$ with $\deg f_i \leq d$ for $i < m$. Then there exists $\gamma, \delta \in \Gamma$ such that for all $\lambda = (\lambda_i)_{i<m} \in K^m$, one has*

$$v(\sum_{i<m} f_i \cdot \lambda_i) \geq \min \left\{ (\min_{i<m, f_i \neq 0} v(\phi^{\deg f_i}(\lambda_i))) + \delta, \gamma \right\} \geq \min \left\{ \phi^d(v(\lambda)) + \delta, \gamma \right\}$$

where $v(\lambda) := \min_{i<m} v(\lambda_i)$. In particular, for every $\gamma' \in \Gamma$, the set

$$\left\{ v(\sum_{i<m} f_i \cdot \lambda_i) \;\middle|\; \lambda = (\lambda_i)_{i<m} \in K^m, v(\lambda) \geq \gamma' \right\}$$

is bounded below by $\min \{ v(\phi^d(\gamma')) + \delta, \gamma \}$.

Proof. Let $i < m$ such that $f_i \neq 0$. Apply Proposition 6.21 to f_i to obtain $\delta_i, \gamma_i \in \Gamma$ such that

$$v(f_i \cdot \mu) \geq \min \left\{ \phi^{\deg f_i}(v(\mu)) + \delta_i, \gamma_i \right\}$$

for all $\mu \in K$. Then for all $\lambda = (\lambda_i)_{i<m} \in K^m$,

$$\begin{aligned} v(\sum_{i<m} f_i \cdot \lambda_i) &\geq \min_{i<m} v(f_i \cdot \lambda_i) \\ &\geq \min_{i<m, f_i \neq 0} \min \left\{ \phi^{\deg f_i}(v(\lambda_i)) + \delta_i, \gamma_i \right\} \\ &\geq \min \left\{ (\min_{i<m, f_i \neq 0} \phi^{\deg f_i}(v(\lambda_i))) + \delta, \gamma' \right\} \end{aligned}$$

with $\delta := \min_{i<m} \delta_i$ and $\gamma' := \min_{i<m} \gamma_i$. Now by Corollary 6.15, there exists a γ_- such that $\phi^j(\tau) \geq \min \{ \phi^d(\tau), \gamma_- \}$ for all $j \leq d$ and $\tau \in \Gamma$. This gives

$$\begin{aligned} \min \left\{ (\min_{i<m, f_i \neq 0} \phi^{\deg f_i}(v(\lambda_i))) + \delta, \gamma' \right\} &\geq \min \left\{ (\min_{i<m} \min \left\{ \phi^d(v(\lambda_i)), \gamma_- \right\}) + \delta, \gamma' \right\} \\ &= \min \left\{ (\min_{i<m} \phi^d(v(\lambda_i))) + \delta, \gamma', \gamma_- + \delta \right\} \\ &= \min \left\{ \phi^d(v(\lambda)) + \delta, \gamma \right\} \end{aligned}$$

for $\gamma = \min \{ \gamma_- + \delta, \gamma' \}$. □

6.23 Assumption. For the rest of the section, K has a weakly valuation independent basis over $\phi(K)$ and $[K : \phi(K)]$ is finite.

6.24 Remark. Every $\phi^n(K)$-linearly independent tuple over K^m is weakly valuation independent over $\phi^n(K)$ by Corollary 6.10 and Lemma 6.11.

6.25 Lemma. *Given n and a basis B of K over $\phi(K)$, there exists $\delta \in \Gamma$ such that for all $\mu \in K$,*

$$(*) \qquad | \, v(\mu) - \phi^n(v((\lambda_b^B(\mu))_{b \in B^n})) \, | \leq \delta \, .$$

(For $\lambda_b^B(\mu)$, see Definition 4.10.)

Proof. With $\omega_b^B = \prod_{k<n} \phi^k(b_k)$ for $b \in B^n$ and $\mu \in K$, we have

$$\mu = \sum_{b \in B^n} \phi^n(\lambda_b^B(\mu))\omega_b^B \, ,$$

so

$$\begin{aligned} v(\mu) &\geq \min_{b \in B^n} v(\phi^n(\lambda_b^B(\mu))\omega_b^B) \\ &= \min_{b \in B^n} v(\phi^n(\lambda_b^B(\mu))) + v(\omega_b^B) \\ &\geq (\min_{b \in B^n} v(\phi^n(\lambda_b^B(\mu)))) + \min_{b \in B^n} v(\omega_b^B) \\ &= \phi^n(v((\lambda_b^B(\mu))_{b \in B^n})) + \delta_1 \end{aligned}$$

for $\delta_1 = \min_{b \in B^n} v(\omega_b^B)$. Because $(\omega_b^B)_{b \in B^n}$ is a basis of K over $\phi^n(K)$, and therefore weakly valuation independent over $\phi^n(K)$, there exists $\delta_2 \in \Gamma$ such that for each tuple $(\mu_b)_{b \in B^n}$ over $\phi^n(K)$, the inequality

$$v(\sum_{b \in B^n} \mu_b \omega_b^B) \leq \min_{b \in B^n} v(\mu_b) + \delta_2$$

holds. Applying this for $\mu_b = \phi^n(\lambda_b^B(\mu))$, one gets

$$\begin{aligned} v(\mu) &\leq \min_{b \in B^n} v(\phi^n(\lambda_b^B(\mu))) + \delta_2 \\ &= \phi^n(v((\lambda_b^B(\mu))_{b \in B^n})) + \delta_2 \end{aligned}$$

Now the choice $\delta = \max\{ -\delta_1, \delta_2 \}$ will make (*) true. □

6.26 Proposition. *Suppose $f_i \in K[\Phi]^m$ for $i < n$ and $(f_i)_{i<n}$ is strongly strongly independent (in the sense of Definition 3.16). Then there exist $\delta, \gamma_0 \in \Gamma$ such that for all $\mu = (\mu_i)_{i<n} \in K^n$,*

$$\min_{i<n} \phi^{\deg f_i}(v(\mu_i)) \geq \min \left\{ v(\sum_{i<n} f_i \cdot \mu_i) + \delta, \gamma_0 \right\} .$$

Proof. Assume first that all f_i have degree equal to $d \geq 1$. Write $f_i = g_i + v_i \Phi^d$ where $g_i \in K[\Phi]^m$ is of degree less than d and $v_i \in K^m$. Then the v_i are $\phi^d(K)$-linearly independent, so they are by Remark 6.24 weakly valuation independent over $\phi^d(K)$ and therefore one can pick $\delta_0 \in \Gamma$ such that for all $\mu = (\mu_i)_{i<n} \in K^n$,

$$v(\sum_{i<n} v_i \phi^d(\mu_i)) \leq v(\phi^d(\mu)) + \delta_0$$

holds, where $\phi^d(\mu) := (\phi^d(\mu_i))_{i<n}$.

By Proposition 6.22, there are $\gamma_1, \delta_1 \in \Gamma$ such that for all $\mu = (\mu_i)_{i<n} \in K^n$, the inequality

$$v(\sum_{i<n} g_i \cdot \mu_i) \geq \min \left\{ v(\phi^{d-1}(\mu)) + \delta_1, \gamma_1 \right\}$$

holds. Apply Lemma 6.14 to find $\gamma_- \in \Gamma$ such that for all $\gamma \in \Gamma$,

$$\gamma < \gamma_- \implies \phi^d(\gamma) + \delta_0 < \phi^{d-1}(\gamma) + \delta_1 \,.$$

Take $\gamma_2 \in \Gamma$ with $\phi^{d-1}(\gamma_2) + \delta_1 \leq \gamma_1$ and set $\gamma_3 = \min \{ \gamma_-, \gamma_2 \}$.

Then for all $\mu = (\mu_i)_{i<n} \in K^n$ with $v(\mu) < \gamma_3$,

$$\begin{aligned} v(\sum_{i<n} v_i \phi^d(\mu_i)) &\leq v(\phi^d(\mu)) + \delta_0 \\ &= \phi^d(v(\mu)) + \delta_0 \\ &< \phi^{d-1}(v(\mu)) + \delta_1 \\ &\leq \min \left\{ v(\phi^{d-1}(\mu)) + \delta_1, \gamma_1 \right\} \leq v(\sum_{i<n} g_i \cdot \mu_i) \,, \end{aligned}$$

so

$$\begin{aligned} v(\sum_{i<n} f_i \cdot \mu_i) &= v((\sum_{i<n} (v_i \Phi^d) \cdot \mu_i) + (\sum_{i<n} g_i \cdot \mu_i)) \\ &= v(\sum_{i<n} v_i \phi^d(\mu_i)) \\ &\leq v(\phi^d(\mu)) + \delta_0 \\ &= \phi^d(v(\mu)) + \delta_0 \,, \end{aligned}$$

i.e.

$$\phi^d(v(\mu)) \geq v(\sum_{i<n} f_i \cdot \mu_i) + \delta$$

for $\delta := -\delta_0$. If $v(\mu) \geq \gamma_3$, then $\phi^d(v(\mu)) \geq \phi^d(\gamma_3)$, so the choice $\gamma_0 = \phi^d(\gamma_3)$ works.

Now we deal with the general case (where not necessarily all f_i have degree equal to d). Fix some $d \geq 1$ that is $\geq \deg f_i$ for all i and a basis B of K over $\phi(K)$, and consider the common

d-enlargement of the f_i with respect to B: Let

$$I := \left\{ (i,b) \mid i < n, b = (b_k)_{k<n-\deg f_i} \in B^{d-\deg f_i} \right\}$$

and for $(i,b) \in I$ with $b = (b_k)_{k<d-\deg f_i}$, let

$$f_{(i,b)} := f_i \prod_{k<d-\deg f_i} b_k \Phi\,.$$

The tuple $(f_{(i,b)})_{(i,b)\in I}$ is strongly independent and all $f_{(i,b)}$ have degree d, so by the special case above there exists $\gamma_1, \delta_1 \in \Gamma$ such that for all $\lambda = (\lambda_{(i,b)})_{(i,b)\in I} \in K^I$,

$$\phi^d(v(\lambda)) \geq \min \left\{ v(\sum_{(i,b)\in I} f_{(i,b)} \cdot \lambda_{(i,b)}) + \delta_1, \gamma_1 \right\}.$$

Given $\mu = (\mu_i)_{i<n} \in K^n$ and applying the above for $\lambda_{(i,b)} = \lambda_b^B(\mu_i)$, $i < n, b \in B^{d-\deg f_i}$, one obtains

$$\phi^d(v(\lambda)) \geq \min \left\{ v(\sum_{i<n} f_i \cdot \mu_i) + \delta_1, \gamma_1 \right\}.$$

Using Lemma 6.25, one can find a $\delta_2 \in \Gamma$ (independent of μ) such that

$$\mid v(\mu_i) - \phi^{d-\deg f_i}(v((\lambda_{(i,b)})_{b\in B^{d-\deg f_i}})) \mid \leq \delta_2\,.$$

This implies

$$\begin{aligned} \phi^{\deg f_i}(v(\mu_i)) &\geq \phi^d(v((\lambda_{(i,b)})_{b\in B^{d-\deg f_i}})) + \phi^{\deg f_i}(\delta_2) \\ &\geq \min \left\{ v(\sum_{i<n} f_i \cdot \mu_i) + \delta_1, \gamma_1 \right\} + \min_{i<n} \phi^{\deg f_i}(\delta_2) \\ &= \min \left\{ v(\sum_{i<n} f_i \cdot \mu_i) + \delta, \gamma_0 \right\} \end{aligned}$$

for $\delta = \delta_1 + \min_{i<n} \phi^{\deg f_i}(\delta_2)$ and $\gamma_0 = \gamma_1 + \min_{i<n} \phi^{\deg f_i}(\delta_2)$. □

6.27 Corollary. *In the same situation as in the last proposition, given γ, there exists γ' such that $v(\sum_{i<n} f_i \cdot \mu_i) \geq \gamma$ implies $v(\mu) \geq \gamma'$, i.e. $v(\mu_i) \geq \gamma'$ for all $i < n$.*

Proof. Let $d = 0$. Apply the previous proposition to find γ_b and $\delta \in \Gamma$ such that

$$v(\mu) < \gamma_b \implies v(\mu) \geq v(\sum_{i<n} f_i \cdot \mu_i) + \delta\,.$$

Given $\gamma \in \Gamma$, set $\gamma' = \min \{ \gamma_b, \gamma - \delta \}$. If $v(\mu) < \gamma'$, then $v(\mu) < \gamma_b$, so $v(\mu) \geq v(\sum_{i<n} f_i \cdot \mu_i) + \delta$,

so

$$v(\sum_{i<n} f_i \cdot \mu_i) + \delta < (\gamma - \delta) + \delta = \gamma\,.$$

Thus, $v(\mu) < \gamma'$ implies $v(\sum_{i<n} f_i \cdot \mu_i) < \gamma$ and therefore $v(\sum_{i<n} f_i \cdot \mu_i) \geq \gamma$ implies $v(\mu) \geq \gamma'$. □

6.28 Definition. 1. Suppose $a \in K$ and $\gamma \in \Gamma_\infty$. Call $V_\gamma(a) := \{\, b \in K \mid v(b-a) \geq \gamma \,\}$ the **ball of radius γ centered at** a.

2. Let $\delta \in \Gamma$ with $\delta \geq 0$, and $S \subseteq K$. A function $f : S \longrightarrow K$ is called **δ-contractive**, if $v(f(a) - f(b)) \geq v(a-b) + \delta$ for all $a, b \in S$.

The proof of the following two lemmas is an easy exercise.

6.29 Lemma. *Suppose $a \in K$, $\gamma \in \Gamma_\infty$, $\delta \in \Gamma$, $\delta \geq 0$ and $f : V_\gamma(a) \longrightarrow V_\gamma(a)$ is δ-contractive. Then*

$$f(V_{\gamma'}(a')) \subseteq V_{\gamma'}(a') \subseteq V_\gamma(a)$$

for $a' = f(a)$ and $\gamma' = \gamma + \delta$.

6.30 Lemma. *Let I be a non-empty index set, $a_i \in K$ and $\gamma_i \in \Gamma_\infty$ for $i \in I$. Assume that $f : \bigcup_{i \in I} V_{\gamma_i}(a_i) \longrightarrow K$ is 0-contractive and $f(V_{\gamma_i}(a_i)) \subseteq V_{\gamma_i}(a_i)$ for all $i \in I$. If $a \in \bigcap_{i \in I} V_{\gamma_i}(a_i)$, then*

$$f(V_\gamma(a)) \subseteq V_\gamma(a) \subseteq V_{\gamma_i}(a_i)$$

for $\gamma = v(f(a) - a)$ and all $i \in I$.

6.31 Lemma. *Let K be maximally valued. Suppose $a \in K$, $\gamma \in \Gamma_\infty$, $\delta \in \Gamma$, $\delta > 0$ and $f : V_\gamma(a) \longrightarrow V_\gamma(a)$ is δ-contractive. Then f has a fixed point.*

Proof. The statement follows by a transfinite induction argument from the previous two lemmas. □

Chapter 7

Asymptotic analysis of pp-sets

In this chapter, let (K, v, Γ) be a valued field (with $v(K^\times) = \Gamma$) and $\Gamma \neq \{ 0 \}$. Let ϕ be a self-embedding of the valued field (K, v, Γ) that has a modulus of growth. We assume that $[K : \phi(K)]$ is finite and that K has a weakly valuation independent basis over $\phi(K)$.

In addition, we fix a set $\mathcal{P}$ of unary predicate symbols such that $V_\infty \in \mathcal{P}$. Every $P \in \mathcal{P}$ is interpreted as an additive subgroup of K. This interpretation is also denoted by P, and V_∞ is interpreted as the zero subgroup. In this chapter, we consider K as a structure for the signature $\sigma_{K[\Phi],\mathcal{P}}$, and accordingly, all formulas are with respect to this signature, and so is the notion of pp-definable.

7.1 Definition. Let I be a finite index set. By a ball in K^I, we mean a set of the form

$$V_\gamma := \left\{ w \in K^I \;\middle|\; v(w) \geq \gamma \right\}$$

for some $\gamma \in \Gamma$ (i.e. a closed ball centered at 0). Note that each ball in K^I is an additive subgroup of the K-vector space K^I.

A subset of K^I is said to be bounded if it is contained in a ball in K^I.

7.1 The large case ($v(x) \to -\infty$)

In this section, each subgroup P with $P \in \mathcal{P}$ is assumed to be bounded. We obtain results about the structure of pp-definable sets in the large, i.e. modulo adding sufficiently large balls.

In this section, X is a finite index set, usually regarded as a set of variables. If Y is a set disjoint from X and $A \subseteq K^{X \cup Y}$, we put $A(w) := \left\{ u \in K^X \;\middle|\; (u, w) \in A \right\}$ for $w \in K^Y$.

7.2 Definition.

1. For subsets $S_1, S_2 \subseteq K^X$, we say that S_1 is **contained** in S_2 **modulo large balls**, if $S_1 + B \subseteq S_2 + B$ for some ball B in K^X , and denote this by $S_1 \overset{\infty}{\subseteq} S_2$. Similarly, call S_1 and S_2 **equal modulo large balls**, if $S_1 + B = S_2 + B$ for some ball B in K^X, and denote this by $S_1 \overset{\infty}{=} S_2$.

2. Let J be a set, $(S_j)_{j \in J}$ be a family of subsets of K^X, and $\delta \in \Gamma$, $\delta \geq 0$. We say that (S_j) has the **δ-maximum property**, if for all $j \in J$ the set $\Theta = \{ v(w) \mid w \in S_j \}$ is empty

or bounded above by $\theta + \delta$ for some $\theta \in \Theta$. We say that (S_j) has the **weak maximum property**, if it has the γ-maximum property for some $\gamma \geq 0$ in Γ. We say that (S_j) has the **weak maximum property modulo large balls**, if there exists a ball B in K^X such that the family $(S_j + B)_{j \in J}$ has the weak maximum property.

3. Let $S \subseteq K^X$ and $\delta \in \Gamma$ with $\delta \geq 0$. We say that S has the **δ-optimal approximation property**, if for all $w \in K^X$ the set $\Theta = \{ v(w' - w) \mid w' \in S \}$ is bounded above by $\theta + \delta$ for some $\theta \in \Theta$. Instead of "0-optimal approximation property" we also say "**optimal approximation property**". We say that S has the **weak optimal approximation property modulo large balls**, if for some ball B in K^X and some $\gamma \geq 0$ in Γ the set $S + B$ has the γ-optimal approximation property.

7.3 Remark. Let $\delta \in \Gamma$, $\delta \geq 0$.

1. The particular notion of ball is not important in defining $\overset{\infty}{\subseteq}$ and $\overset{\infty}{=}$, i.e., if $\mathcal{V}$ is any collection of subgroups of K^X such that every element contained in $\mathcal{V}$ is contained in a ball in K^X and vice versa, then one can replace the notion of a ball in K^X by being an element of $\mathcal{V}$.

2. $S \subseteq K^X$ has the δ-optimal approximation property if and only if S is non-empty and the family $(S + w)_{w \in K^X}$ has the δ-maximum property.

3. Let Y be a finite set disjoint from X and A an additive subgroup of the product group $K^{X \cup Y}$. The family $(A(w))_{w \in K^Y}$ has the δ-maximum property if and only if $A(0)$ has the δ-optimal approximation property. This is the case, since for every $w \in K^Y$ the set $A(w)$ is empty or a coset of $A(0)$.

4. Suppose the family $(S_j)_{j \in J}$ of subsets of K^X has the δ-maximum property, and B is a ball in K^X. Then the family $(S_j + B)_{j \in J}$ has the δ-maximum property: Let $\gamma \in \Gamma$ such that $B = V_\gamma$. Let $j \in J$ and set $\Theta := \{ v(w) \mid w \in S_j \}$, $\Theta' := \{ v(w) \mid w \in S_j + B \}$. If $\gamma \leq \theta$ for some $\theta \in \Theta$, then $\infty \in \Theta'$. If $\gamma > \theta$ for all $\theta \in \Theta$, then $\Theta' = \Theta$.

5. Suppose $S \subseteq K^X$ has the δ-optimal approximation property and B is a ball in K^X. Then $S + B$ has the δ-optimal approximation property.

6. If $\Gamma = \mathbb{Z}$, then every non-empty subset of K^X has the weak optimal approximation property modulo large balls.

7.4 Lemma. *Let I be a finite set, $T \in K[\Phi]^{I \times X}$ and $P \in \mathcal{P}^I$. The following are equivalent:*

1. *The subgroup of K^X defined by $T \cdot X \in P$ is bounded.*

2. *There exists a product E of restricted elementary matrices in $\mathrm{MAT}_X(K[\Phi])$ such that the columns of TE are strongly independent.*

Proof. 2 $\Longrightarrow$ 1: Let E be a product of restricted elementary matrices in $\mathrm{MAT}_X(K[\Phi])$ such that the columns of TE are strongly independent. Because each P_i is bounded (by the assumption that the subgroups in $\mathcal{P}$ are bounded), Corollary 6.27 yields that the subgroup B of K^X defined by $(TE) \cdot X \in P$ is bounded. Thus the image B' of B under the map $K^X \longrightarrow K^X$, $v \mapsto E \cdot v$ is bounded. Note that B' is the subgroup defined by $T \cdot X \in P$.

1 $\Longrightarrow$ 2: By Proposition 3.18, there exists a product E of restricted elementary matrices in $\mathrm{MAT}_X(K[\Phi])$ such that the non-zero columns of TE are strongly independent. Suppose there exists a zero column in TE, say with index $x_0 \in X$. Define $f \in K[\Phi]^X$ by $f(x) := E(x, x_0)$ for $x \in X$. Then $f \cdot K$ is unbounded, because Γ is non-trivial. Since $f \cdot K$ is contained in the solution set of $T \cdot X \in P$, this solution set is unbounded. □

7.5 Proposition. *Let $A \subseteq K^X$ be a pp-definable subgroup. Then there exist a finitely generated right submodule M of $K[\Phi]^X$, a finite set of variables Y disjoint from X, a finite set I, an element $P \in \mathcal{P}^I$ and a matrix $T \in K[\Phi]^{I\times(X\cup Y)}$ such that the subgroup of $K^{X\cup Y}$ defined by $T\cdot(X \cup Y) \in P$ is bounded and for the subgroup B of K^X defined by the formula $\exists Y\,(T \cdot (X \cup Y) \in P)$, we have*

$$A = B + M \cdot K\,.$$

For any M, Y, I, P, T, B as above, B is bounded and $A \overset{\infty}{=} M\cdot K$. If A is bounded, then A is the image under the projection map $K^{X\cup Y} \longrightarrow K^X$ of a bounded subgroup of $K^{X\cup Y}$ defined by a conjunction of atomic formulas.

Therefore, the study of pp-definable sets in K^X up to (adding) large balls amounts to studying sets of the form $M \cdot K$ for finitely generated $K[\Phi]$-modules $M \subseteq K[\Phi]^X$.

Proof. By Remark 5.11, there exist a finite set of variables Z disjoint from X, a finite set J, a tuple $Q \in \mathcal{P}^J$ and a matrix $S \in K[\Phi]^{J\times(X\cup Z)}$ such that A is defined by the formula $\exists Z\,(S\cdot(X\cup Z) \in Q)$. Set $H = X \cup Z$. By Proposition 3.18, there exists a (possibly empty) product E of restricted elementary matrices in $\mathrm{MAT}_H(K[\Phi])$ such that the non-zero column vectors of SE are strongly independent. Let $H_0 \subseteq H$ be the set of indices of zero columns and $H_{\mathrm{ind}} \subseteq H$ be the set of indices of non-zero columns of SE. (The subscript "ind" stands for "strongly independent".)

Now take as M the submodule of $K[\Phi]^X$ generated by the columns of $E{\restriction}_{X\times H_0}$. Take a bijection $h : H_{\mathrm{ind}} \longrightarrow Y$ where Y is disjoint from X. To keep notations simple we pretend that $Y = H_{\mathrm{ind}}$ and that h is the identity on Y. Set $I := X\dot\cup J$ and define $P = (P_i)_{i\in I} \in \mathcal{P}^I$ by $P{\restriction}_J = Q$ and $P_x = V_\infty$ for $x \in X$, and define $T \in K[\Phi]^{I\times(X\dot\cup Y)}$ by $T{\restriction}_{X\times X} = -\mathrm{Id}_X$, $T{\restriction}_{X\times Y} = E{\restriction}_{X\times Y}$, $T{\restriction}_{J\times X} = 0$ and $T{\restriction}_{J\times Y} = (SE){\restriction}_{J\times Y}$.

Here is a picture of $T \cdot (X \dot\cup Y) \in P$:

$$\begin{array}{c} X \\ \\ J \end{array} \begin{array}{|c|c|} \hline -I & E\restriction_{X\times Y} \\ \hline 0 & (SE)\restriction_{J\times Y} \\ \hline \end{array} \quad \begin{array}{|c|} \hline X \\ \hline Y \\ \hline \end{array} \quad \in \quad \begin{array}{c} \{\,0\,\}^X \\ \times \\ \prod_{j\in J} Q_j \end{array}$$

$$X \qquad Y \simeq H_{\mathrm{ind}}$$

We claim that the first part of the proposition holds with these choices of M, Y, I, P, T. The following proves this claim. Let $C_0 = K^{H_0}$ and D_{ind} be the subgroup of $K^{H_{\mathrm{ind}}}$ defined by the formula $(SE)\restriction_{J\times H_{\mathrm{ind}}} \cdot Y \in Q$ (recall that the set Y is identified with H_{ind}). Then with $C := \{\,0\,\}^{H_{\mathrm{ind}}} \times C_0 \subseteq K^H$ and $D := D_{\mathrm{ind}} \times \{\,0\,\}^{H_0} \subseteq K^H$, the solution set of $(SE) \cdot (X \cup Z) \in Q$ is equal to $C + D$. Therefore, the solution set of $S \cdot (X \cup Z) \in Q$ is equal to $C' + D'$ for $C' = \{\, E \cdot w \mid w \in C \,\} \subseteq K^H$ and $D' = \{\, E \cdot w \mid w \in D \,\} \subseteq K^H$. Furthermore, let C'' and D'' be the images of C' and D' respectively under the projection map $K^H \longrightarrow K^X$. Then $A = C'' + D''$. Note that $C' = E\restriction_{H\times H_0} \cdot K^{H_0}$ and $C'' = E\restriction_{X\times H_0} \cdot K^{H_0}$, so $C'' = M \cdot K$. The set D'' is defined by the formula

$$\exists (Y \cup H_0)\,(\mathrm{Id}_X \cdot X = E\restriction_{X\times H} \cdot (Y \cup H_0) \wedge (Y \cup H_0) \in D)\,,$$

and because all elements of D are zero in the components H_0, this has the same solutions as

$$\exists Y\,(\mathrm{Id}_X \cdot X = E\restriction_{X\times H_{\mathrm{ind}}} \cdot Y \wedge (SE)\restriction_{J\times H_{\mathrm{ind}}} \cdot Y \in Q)\,,$$

which is equivalent to

$$\exists Y\,(T \cdot (X \dot\cup Y) \in P)\,.$$

Observe that the matrix $E' \in \mathrm{MAT}_{X\cup Y}(K[\Phi])$ given by $E'\restriction_{X\times X} = \mathrm{Id}_X$, $E'\restriction_{Y\times Y} = \mathrm{Id}_Y$ $E'\restriction_{X\times Y} = E\restriction_{X\times Y}$, $E'\restriction_{Y\times X} = 0$, is a product of restricted elementary matrices and $TE' = (-\mathrm{Id}_X) \sqcup (SE)\restriction_{J\times Y}$ has strongly independent columns. By Lemma 7.4, the subgroup of $K^{X\cup Y}$ defined by $T \cdot (X \dot\cup Y) \in P$ is bounded.

Suppose that M, Y, I, P, T, B are as in the first part of the proposition. Clearly B is bounded and thus $A \overset{\infty}{=} M \cdot K$. Now suppose that A is bounded. Since the valuation on K is non-trivial, M must be $\{\,0\,\}$, so A is the projection of the bounded subgroup defined by the formula $T \cdot (X \dot\cup Y) \in P$. ☐

7.6 Example. Consider the pp-definable subsets $S_1 = \Phi \cdot K$ and $S_2 = (\Phi - 1) \cdot K$ of K. Then also $S_1 \cap S_2$ is pp-definable, so by the proposition there exists a finitely generated submodule M

of $K[\Phi]$ with $S_1 \cap S_2 \overset{\infty}{\cong} M \cdot K$. Here we show how to compute an element of $K[\Phi]$ that generates such a module M. We have $x \in S_1 \cap S_2$ if and only if

$$\exists z_1, z_2 \ \begin{pmatrix} 1 & -\Phi & 0 \\ 1 & 0 & 1-\Phi \end{pmatrix} \cdot \begin{pmatrix} x \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

Also

$$\begin{pmatrix} 1 & -\Phi & 0 \\ 1 & 0 & 1-\Phi \end{pmatrix} \begin{pmatrix} 1 & \Phi-\Phi^2 & \Phi \\ 0 & 1-\Phi & 1 \\ 0 & -\Phi & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} ,$$

where

$$\begin{pmatrix} 1 & \Phi-\Phi^2 & \Phi \\ 0 & 1-\Phi & 1 \\ 0 & -\Phi & 1 \end{pmatrix} = \begin{pmatrix} 1 & \Phi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\Phi & 1 \end{pmatrix}$$

is a product of restricted elementary matrices and the first and third column of $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ are strongly independent. So we get

$$S_1 \cap S_2 \overset{\infty}{\cong} \pi_1(\begin{pmatrix} \Phi-\Phi^2 \\ 1-\Phi \\ -\Phi \end{pmatrix} \cdot K) = (\Phi - \Phi^2) \cdot K$$

where π_1 denotes the projection on the first component.

Now assume that $t \in K \setminus \phi(K)$ and consider the pp-definable subsets $S_1 = \Phi^2 \cdot K$ and $S_2 = (\Phi^2 - t\Phi) \cdot K$ of K. We will show that $S_1 \cap S_2 \overset{\infty}{\cong} \{0\}$, so $S_1 \cap S_2$ is bounded. Again, we have $x \in S_1 \cap S_2$ if and only if

$$\exists z_1, z_2 \ \begin{pmatrix} 1 & -\Phi^2 & 0 \\ 1 & 0 & t\Phi-\Phi^2 \end{pmatrix} \cdot \begin{pmatrix} x \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

Also

$$\begin{pmatrix} 1 & -\Phi^2 & 0 \\ 1 & 0 & t\Phi-\Phi^2 \end{pmatrix} \begin{pmatrix} 1 & \Phi^2 & \Phi^2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \Phi^2 & t\Phi \end{pmatrix} ,$$

where

$$\begin{pmatrix} 1 & \Phi^2 & \Phi^2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \Phi^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is a product of restricted elementary matrices and the columns of $\begin{pmatrix} 1 & 0 & 0 \\ 1 & \Phi^2 & t\Phi \end{pmatrix}$ are strongly independent. So we obtain $S_1 \cap S_2 \overset{\infty}{\cong} \{0\}$.

The following lemma provides a uniform bound for the non-empty sections over a bounded set of a family defined by a conjunction of atomic formulas. It is related to the last part of the previous proposition.

7.7 Lemma. *Suppose X and Z are disjoint finite sets of variables, J a finite set, $P \in \mathcal{P}^J$ and $S \in K[\Phi]^{J \times (X \cup Z)}$.*

Let B be a ball in K^X. Then there exists a ball B' in K^Z such that for all $w \in B$: If there exists $u \in K^Z$ such that $K \models S \cdot (w \frown u) \in P$, then there exists $u' \in B'$ such that $K \models S \cdot (w \frown u') \in P$.

Proof. We assume that J and X are disjoint. Let C be a ball in K such that $B = C^X$. Without loss of generality, we may assume that $\mathcal{P}$ contains a predicate for C (also denoted by C).

Put $H := X \cup Z$ and $J' := J \cup X$. Define $P' \in \mathcal{P}^{J'}$ by $P'\restriction_J = P$ and $P'_x = C$ for $x \in X$. Define $S' \in K[\Phi]^{J' \times H}$ by $S'\restriction_{J \times H} = S$, $S'\restriction_{X \times X} = \mathrm{Id}_X$ and $S'\restriction_{X \times Z} = 0$.

By Proposition 3.18, there exists a (possibly empty) product E of restricted elementary matrices in $\mathrm{MAT}_H(K[\Phi])$ such the non-zero column vectors of $S'E$ are strongly independent. Let $H_0 \subseteq H$ be the set of indices of zero columns and $H_{\text{ind}} \subseteq H$ be the set of indices of non-zero columns of $S'E$.

Consider the following additive subgroups of K^H:

$$
\begin{aligned}
A &= \{\, b \in K^H \mid K \models S' \cdot b \in P' \,\}\ , \\
A_{\text{ind}} &= E \cdot (\{\, a \in K^{H_{\text{ind}}} \mid K \models S'E\restriction_{J' \times H_{\text{ind}}} \cdot a \in P' \,\} \times \{\,0\,\}^{H_0})\,, \\
A_0 &= E \cdot (\{\,0\,\}^{H_{\text{ind}}} \times K^{H_0})\,.
\end{aligned}
$$

Note that $A = \{\, (w, u) \in B \times K^Z \mid K \models S \cdot (w^\frown u) \in P \,\}$. We have $A = A_{\text{ind}} + A_0$. Let $\pi : K^H \longrightarrow K^X$ denote the canonical projection. Then $\pi(A) = \pi(A_{\text{ind}}) + \pi(A_0)$. Note that $\pi(A_0)$ is unbounded or equal to $\{\,0\,\}$, because Γ is non-trivial and $\pi(A_0)$ is an image under a term map. Because $\pi(A)$ is bounded, we get $\pi(A) = \pi(A_{\text{ind}}) \subseteq B$ and $\pi(A_0) = \{\,0\,\}$.

We have that A_{ind} is bounded, since the non-zero columns of $S'E$ are strongly independent. Now it is clear that choosing as B' any ball in K^Z such that $A_{\text{ind}} \subseteq B \times B'$ will satisfy the conclusion of the lemma. □

7.8 Lemma. *Given any non-empty X, the following conditions are equivalent:*

1. *For every finitely generated submodule M of $K[\Phi]$, the set $M \cdot K \subset K$ has the weak optimal approximation property modulo large balls.*
2. *For every finitely generated submodule M of $K[\Phi]^X$, the set $M \cdot K$ has the weak optimal approximation property modulo large balls.*
3. *Every pp-definable set in K^X has the weak optimal approximation property modulo large balls.*
4. *For each finite set Y disjoint from X and each pp-definable set $A \subseteq K^{X \cup Y}$, the family $(A(w))_{w \in K^Y}$ has the weak maximum property modulo large balls.*
5. *For each finite set Y disjoint from X and each $A \subseteq K^{X \cup Y}$ defined by a pp-formula with parameters from K, the family $(A(w))_{w \in K^Y}$ has the weak maximum property modulo large balls.*

Proof. The implication $2 \implies 3$ follows from Proposition 7.5 and Remark 7.3, part 5. The implication $3 \implies 4$ follows from Remark 7.3, part 3. The implication $4 \implies 5$ is trivial. The implication $5 \implies 2$ follows from Remark 7.3, part 2. The implication $2 \implies 1$ is trivial. Last, we show the implication $1 \implies 2$. We may assume without loss of generality that $X = B^n$ where B is a basis of K over $\phi(K)$.

Let $(g_j)_{j\in J}$ be a finite tuple of elements of $K[\Phi]^X$, and put $f_{(j,b)} := q_b^B g_j(b) \in K[\Phi]$ for $j \in J$ and $b \in B^n$ where $q_b^B \in K[\Phi]$ is the basis polynomial with respect to B and b. Suppose condition 1 holds; so $\sum_{(j,b)\in J\times B^n} f_{(j,b)} \cdot K$ has the weak optimal approximation property modulo large balls. Then by Lemma 6.25, the set $\sum_{j\in J} g_j \cdot K$ has the weak optimal approximation property modulo large balls. □

In the rest of this section, we set $N := K[\Phi]^X$, and consider N as a right $K[\Phi]$-module.

7.9 Lemma. *Let $(f_i)_{i<m} \in N^m$ and $(g_j)_{j<n} \in N^n$. Assume that the 1-enlargement space of each g_j is contained in the (right) K-linear span of $\{ f_i \mid i < m \} \cup \{ g_k \mid k < n \}$. Then there exists $\delta \in \Gamma$ such that for all $(\lambda_j)_{j<n} \in K^n$, there exists $(\mu_j)_{j<n} \in K^n$ with the property*

$$\phi(v((\mu_j)_{j<n})) \geq v((\lambda_j)_{j<n}) + \delta \text{ and } \sum_{j<n} g_j \cdot \lambda_j \in \sum_{j<n} g_j \cdot \mu_j + \sum_{i<m} f_i \cdot K \subseteq K^X .$$

Proof. Pick a basis B of K over $\phi(K)$. Since the 1-enlargement space of each g_j is contained in the (right) K-linear span of all the f_i and g_j, one can find $C_{ijb}, D_{kjb} \in K$ for $i < m, j < n, k < n$ and $b \in B$ such that for each j and b,

$$g_j b\Phi = \sum_{i<m} f_i C_{ijb} + \sum_{k<n} g_k D_{kjb} .$$

Fix $(\lambda_j)_{j<n} \in K^n$. Take $\mu_{jb} \in K$ for $j < n$ and $b \in B$ such that $\lambda_j = \sum_{b\in B} (b\Phi) \cdot \mu_{jb}$ for each $j < n$. Then

$$\begin{aligned}
\sum_{j<n} g_j \cdot \lambda_j &= \sum_{j<n, b\in B} (g_j b\Phi) \cdot \mu_{jb} \\
&= \sum_{j<n, b\in B} \Big((\sum_{i<m} f_i C_{ijb}) \cdot \mu_{jb} + (\sum_{k<n} g_k D_{kjb}) \cdot \mu_{jb} \Big) \\
&= \sum_{i<m} f_i \cdot (\sum_{j<n, b\in B} C_{ijb}\mu_{jb}) + \sum_{k<n} g_k \cdot (\sum_{j<n, b\in B} D_{kjb}\mu_{jb}) \\
&= \sum_{i<m} f_i \cdot (\sum_{j<n, b\in B} C_{ijb}\mu_{jb}) + \sum_{k<n} g_k \cdot \mu_k
\end{aligned}$$

with $\mu_k := \sum_{j<n, b\in B} D_{kjb}\mu_{jb}$. Observe that

$$\sum_{i<m} f_i \cdot (\sum_{j<n, b\in B} C_{ijb}\mu_{jb}) \in \sum_{i<m} f_i \cdot K ,$$

which yields the desired inclusion in the lemma. Next, note that $v((\mu_j)_{j<n}) \geq \delta_1 + v((\mu_{jb})_{j<n, b\in B})$ for $\delta_1 = \min_{i<m, j<n, b\in B} v(D_{ijb})$. Hence $\phi(v((\mu_j)_{j<n})) \geq \phi(\delta_1) + \phi(v((\mu_{jb})_{j<n, b\in B}))$. By Lemma 6.25, there exists $\delta_2 \in \Gamma$ independent of $(\lambda_j)_{j<n}$ such that $\phi(v((\mu_{jb})_{b\in B})) \geq v(\lambda_j) + \delta_2$, so

$$\phi(v((\mu_j)_{j<n})) \geq v((\lambda_j)_{j<n}) + (\phi(\delta_1) + \delta_2) .$$

□

A problem with the last lemma is that later we would like to choose $(g_j)_{j<n}$ as a strongly independent generating tuple of some submodule M of N, and $(f_i)_{i<m}$ should induce a basis of

$$(M \cap N_{\leq d})/(M \cap N_{<d})$$

for some $d \geq \max_{j<n} \deg g_j$. In this situation, we have to relax the condition that the enlargement of g_j is in the K-linear span of the g_j and f_i to also allow multiplying the g_j with certain elements of $K[\Phi]$. This is stated in the following lemma, whose proof proceeds roughly as the proof of the previous lemma, with slightly more involved notation.

7.10 Lemma. *Let $d \in \mathbb{N}$, $(f_i)_{i<m} \in N^m$ and $(g_j)_{j<n} \in N^n$. Assume that for all $j < n$, we have $g_j \neq 0$ and $d_j := \deg g_j \leq d$. Also assume that for each $j < n$, the $(d - d_j)$-enlargement space of g_j is contained in the (right) K-linear subspace of N generated by*

$$\{\, f_i \mid i < m \,\} \cup \bigcup_{k<n, l<d-d_k} l\text{-enlargement space of } g_k\,.$$

For a tuple $(\lambda_j)_{j<n} \in K^n$, define

$$\hat{v}((\lambda_j)_{j<n}) = \min_{j<n} \phi^{d_j}(v(\lambda_j))\,.$$

Then there exist $\delta, \gamma \in \Gamma$ such that for all $(\lambda_j)_{j<n} \in K^n$, there exists $(\mu_j)_{j<n} \in K^n$ with the property

$$\phi(\hat{v}((\mu_j)_{j<n})) \geq \min\{\, \hat{v}((\lambda_j)_{j<n})) + \delta, \gamma \,\} \text{ and } \sum_{j<n} g_j \cdot \lambda_j \in \sum_{j<n} g_j \cdot \mu_j + \sum_{i<m} f_i \cdot K \subseteq K^X\,.$$

Proof. Pick a basis B of K over $\phi(K)$. Let $j < n$. By the hypothesis on g_j, one can find $C_{ijb} \in K$, $D_{kjb} \in K[\Phi]_{<d-d_k}$ for $i < m, k < n, b \in B^{d-d_j}$ such that

$$g_j q_b^B = \sum_{i<m} f_i C_{ijb} + \sum_{k<n} g_k D_{kjb}\,,$$

where q_b^B is the basis polynomial with respect to B and b (see Definition 4.10). For $h \in K[\Phi]$, one can find $\delta_h \in \Gamma$ such that $v(h \cdot \lambda) \geq \delta_h + \min_{i \leq \deg h} \phi^i(v(\lambda))$ for all $\lambda \in K$. Choose $\delta_1 \in \Gamma$ such that $\delta_1 \leq \delta_h$ for all $h = D_{kjb}$, $j < n, k < n, b \in B^{d-d_j}$. By Corollary 6.15, we choose $\delta_2 \in \Gamma$ such that for all $\gamma' \in \Gamma$ and all i_0, i_1 with $i_0 \leq i_1 \leq d$, one has

$$\phi^{i_0}(\gamma') \geq \min\left\{\, \phi^{i_1}(\gamma'), \delta_2 \,\right\}\,.$$

Fix $(\lambda_j)_{j<n} \in K^n$. For $j < n$, let $(\mu_{jb})_{b \in B^{d-d_j}} \in K^{B^{d-d_j}}$ be the unique tuple such that $\lambda_j = \sum_{b \in B^{d-d_j}} q_b^B \cdot \mu_{jb}$. Then by Lemma 6.25, there exists $\delta_3 \in \Gamma$ independent of (λ_j) such that for all $j < n$ and $b \in B^{d-d_j}$,

$$\phi^{d-d_j}(v(\mu_{jb})) \geq v(\lambda_j) + \delta_3 \,.$$

We have

$$\begin{aligned}
\sum_{j<n} g_j \cdot \lambda_j &= \sum_{j<n, b\in B^{d-d_j}} (g_j q_b^B) \cdot \mu_{jb} \\
&= \sum_{j<n, b\in B^{d-d_j}} \big((\sum_{i<m} f_i C_{ijb}) \cdot \mu_{jb} + (\sum_{k<n} g_k D_{kjb}) \cdot \mu_{jb} \big) \\
&= \sum_{i<m} f_i \cdot (\sum_{j<n, b\in B^{d-d_j}} C_{ijb} \mu_{jb}) + \sum_{k<n} g_k \cdot (\sum_{j<n, b\in B^{d-d_j}} D_{kjb} \cdot \mu_{jb}) \\
&= \sum_{i<m} f_i \cdot (\sum_{j<n, b\in B^{d-d_j}} C_{ijb} \mu_{jb}) + \sum_{k<n} g_k \cdot \mu_k
\end{aligned}$$

with $\mu_k := \sum_{j<n, b\in B^{d-d_j}} D_{kjb} \cdot \mu_{jb} \in K$ for $k < n$. Observe that

$$\sum_{i<m} f_i \cdot (\sum_{j<n, b\in B^{d-d_j}} C_{ijb} \mu_{jb}) \in \sum_{i<m} f_i \cdot K \,,$$

which yields the desired inclusion in the lemma. Next, let $k < n$, and note that

$$v(\mu_k) \geq \delta_1 + \min_{j<n, b\in B^{d-d_j}, i<d-d_k} \phi^i(v(\mu_{jb})) \,,$$

so

$$\begin{aligned}
\phi(\phi^{d_k}(v(\mu_k))) &\geq \phi^{d_k+1}\big(\delta_1 + \min_{j<n, b\in B^{d-d_j}, i<d-d_k} \phi^i(v(\mu_{jb}))\big) \\
&= \phi^{d_k+1}(\delta_1) + \min_{j<n, b\in B^{d-d_j}, i<d-d_k} \phi^{d_k+1}(\phi^i(v(\mu_{jb}))) \\
&= \phi^{d_k+1}(\delta_1) + \min_{j<n, b\in B^{d-d_j}, d_k+1\leq i\leq d} \phi^i(v(\mu_{jb})) \\
&\geq \phi^{d_k+1}(\delta_1) + \min_{j<n, b\in B^{d-d_j}} \min \Big\{ \phi^d(v(\mu_{jb})) \,,\, \delta_2 \Big\} \\
&= \phi^{d_k+1}(\delta_1) + \min_{j<n, b\in B^{d-d_j}} \min \Big\{ \phi^{d_j}(\phi^{d-d_j}(v(\mu_{jb}))) \,,\, \delta_2 \Big\} \\
&\geq \phi^{d_k+1}(\delta_1) + \min_{j<n, b\in B^{d-d_j}} \min \Big\{ \phi^{d_j}(v(\lambda_j) + \delta_3) \,,\, \delta_2 \Big\} \\
&= \phi^{d_k+1}(\delta_1) + \min_{j<n} \min \Big\{ \phi^{d_j}(v(\lambda_j) + \delta_3) \,,\, \delta_2 \Big\} \\
&\geq \min \Big\{ \phi^{d_k+1}(\delta_1) + \min_{j<n} \phi^{d_j}(\delta_3) + \min_{j<n} \phi^{d_j}(v(\lambda_j)) \,,\, \delta_2 + \phi^{d_k+1}(\delta_1) \Big\} \\
&\geq \min \{ \, \delta + \hat{v}((\lambda_j)_{j<n}) \,,\, \gamma \, \}
\end{aligned}$$

with $\delta = \min_{j<n} \phi^{d_j+1}(\delta_1) + \min_{j<n} \phi^{d_j}(\delta_3)$ and $\gamma = \delta_2 + \min_{j<n} \phi^{d_j+1}(\delta_1)$. It follows that

$$\phi(\hat{v}((\mu_j)_{j<n})) = \min_{j<n} \phi(\phi^{d_j}(v(\mu_j))) \geq \min \{ \delta + \hat{v}((\lambda_j)_{j<n}), \gamma \} .$$

□

Below, the subscripts "h" and "l" stand for "high" and "low".

7.11 Lemma. *Suppose M is a finitely generated submodule of N, and $d \in \mathbb{N}$ is greater than or equal to the degrees of the elements of some generating set of M. Let $M_h \subseteq M \cap N_{\leq d}$ be a K-subspace such that $M \cap N_{\leq d}$ is contained in $M_h + N_{<d}$. Also let $M_l = M \cap N_{<d}$.*

1. *There exist $\delta, \gamma \in \Gamma$ such that for all $w \in M_l \cdot K$ there exists $w' \in M_l \cdot K$ with $w - w' \in M_h \cdot K$ and $\phi(v(w')) \geq \min \{ v(w) + \delta, \gamma \}$.*
2. *If $M_h \cdot K \cap M_l \cdot K$ has the weak optimal approximation property modulo large balls, then $M \cdot K \overset{\infty}{\subseteq} M_h \cdot K$.*

Proof. We start proving the first statement. Pick $f_i \in N$ for $i < m$ such that they generate M_h as a K-subspace. Also, using Proposition 4.12, pick a strongly independent tuple $(g_j)_{j<n} \in N^n$ that generates the same submodule of N as M_l. By Proposition 4.12, part 1, every g_j has degree $\leq d - 1$, so $g_j \in M_l$.

Claim. The hypothesis of Lemma 7.10 is satisfied for the above chosen f_i and g_j.

Let $j < n$, and g be any element of the $(d - \deg g_j)$-enlargement space of g_j. Then g lies in the submodule generated by M_l, so also in M, and has degree $\leq d$. By the assumption on M_h, there exists $f \in M_h$ (so f is a K-linear combination of the f_i) such that $g - f$ has degree $\leq d - 1$. Since $f - g \in M$, and therefore $f - g \in M_l$, it follows by Proposition 4.12, part 2, that the element $f - g$ lies in the K-linear span of

$$\bigcup_{k<n, l<d-\deg g_k} l\text{-enlargement space of } g_k .$$

This proves the claim.

Let $w \in M_l \cdot K$. Then there exist $\mu_j \in K$ for $j < n$ such that $w = \sum_{j<n} g_j \cdot \mu_j$. By Lemma 7.10, there exist $\delta_0, \gamma_0 \in \Gamma$ (independent of the μ_j and therefore independent of w) and $\mu'_j \in K$ for $j < n$ such that setting $w' = \sum_{j<n} g_j \cdot \mu'_j$ one has $w - w' \in M_h \cdot K$ and

$$\phi(\hat{v}((\mu'_j)_{j<n})) \geq \min \{ \hat{v}((\mu_j)_{j<n})) + \delta_0, \gamma_0 \}$$

where $\hat{v}((\lambda_j)_{j<n})$ is defined to be $\min_{j<n} \phi^{\deg g_j}(v(\lambda_j))$ for all $(\lambda_j)_{j<n} \in K^n$. Because the g_j are strongly independent, Proposition 6.26 ensures the existence of $\delta_1, \gamma_1 \in \Gamma$ (independent of the μ_j)

such that

$$\hat{v}((\mu_j)_{j<n}) = \min_{j<n} \phi^{\deg g_j}(v(\mu_j)) \geq \min\left\{ v(\sum\nolimits_{j<n} g_j \cdot \mu_j) + \delta_1, \gamma_1 \right\} = \min\{\, v(w) + \delta_1, \gamma_1 \,\} .$$

By Proposition 6.22, there exist $\delta_2, \gamma_2 \in \Gamma$ (independent of the μ'_j) such that

$$v(w') = v(\sum_{j<n} g_j \cdot \mu'_j) \geq \min\left\{ (\min_{j<n} v(\phi^{\deg g_j}(\mu'_j))) + \delta_2, \gamma_2 \right\} = \min\{\, \hat{v}((\mu'_j)_{j<n}) + \delta_2, \gamma_2 \,\} .$$

Assembling the previous statements, one gets

$$\begin{aligned}
\phi(v(w')) &\geq \phi(\min\{\, \hat{v}((\mu'_j)_{j<n}) + \delta_2 \,,\, \gamma_2 \,\}) \\
&= \min\{\, \phi(\hat{v}((\mu'_j)_{j<n})) + \phi(\delta_2) \,,\, \phi(\gamma_2) \,\} \\
&\geq \min\{\, \min\{\, \hat{v}((\mu_j)_{j<n}) + \delta_0 \,,\, \gamma_0 \,\} + \phi(\delta_2) \,,\, \phi(\gamma_2) \,\} \\
&\geq \min\{\, \min\{\, \min\{\, v(w) + \delta_1 \,,\, \gamma_1 \,\} + \delta_0 \,,\, \gamma_0 \,\} + \phi(\delta_2) \,,\, \phi(\gamma_2) \,\} \\
&= \min\{\, v(w) + \delta \,,\, \gamma \,\}
\end{aligned}$$

for $\gamma = \min\{\, \phi(\gamma_2), \gamma_0 + \phi(\delta_2), \gamma_1 + \delta_0 + \phi(\delta_2) \,\}$ and $\delta = \delta_1 + \delta_0 + \phi(\delta_2)$.

To prove the second statement, let $S = M_h \cdot K \cap M_l \cdot K$ and assume that S has the weak optimal approximation property modulo large balls. So there exist a ball $B' \subseteq K^X$ and $\delta' \in \Gamma$ with $\delta' \geq 0$ such that $S + B'$ has the δ'-optimal approximation property. Now let $x \in M \cdot K$ and consider the set $S' = \{\, w \in M_l \cdot K \mid x - w \in M_h \cdot K \,\}$. This set is non-empty, because $M \cdot K = M_l \cdot K + M_h \cdot K$, and therefore is a coset of the subgroup S of K^X. So $S' + B'$ has the δ'-optimal approximation property.

Pick $w \in S'$ and $y \in B'$ such that for all $u \in S' + B'$ one has $v(u) \leq v(w + y) + \delta'$. Because $w \in M_l \cdot K$, there exists by the first part $w' \in M_l \cdot K$ such that $w - w' \in M_h \cdot K$ and

$$\phi(v(w')) \geq \min\{\, v(w) + \delta, \gamma \,\} .$$

We have $w' \in S'$, because $w' \in M_l \cdot K$ and $x - w' = (x - w) + (w - w') \in M_h \cdot K$. So $v(w') \leq v(w + y) + \delta'$. Assume first that $w \notin B'$. Then $v(w + y) = v(w)$, so

$$v(w') \leq v(w) + \delta' .$$

This yields

$$\phi(v(w')) \leq \phi(v(w)) + \phi(\delta') ,$$

so

$$\min\{\, v(w) + \delta, \gamma \,\} \leq \phi(v(w)) + \phi(\delta') .$$

By Lemma 6.16, the set $\left\{\, u \in K^X \;\middle|\; \min\{\, v(u) + \delta, \gamma \,\} \leq \phi(v(u)) + \phi(\delta') \,\right\}$ is contained in a ball B''

in K^X. Then for $B = B' \cup B''$, we have $w \in B$, so $x \in w + M_h \cdot K \subseteq B + M_h \cdot K$. $\square$

7.12 Lemma. *Suppose I is some finite index set with disjoint subsets I_1 and I_2 and $(f_i)_{i\in I}$ a strongly independent tuple of elements in N. Then for any $\gamma \in \Gamma$, the subset*

$$(\sum_{i\in I_1} f_i \cdot K + V_\gamma) \cap (\sum_{i\in I_2} f_i \cdot K + V_\gamma)$$

of K^X is bounded.

Proof. Let $\gamma \in \Gamma$. By Corollary 6.27, there exists $\gamma' \in \Gamma$ such that

$$v(\sum_{i\in I} f_i \cdot \lambda_i) \geq \gamma \implies v(\lambda) \geq \gamma'$$

for all $\lambda = (\lambda_i)_{i\in I} \in K^I$. By Lemma 6.19, part 3, there exists $\delta \in \Gamma$ such that $v(\sum_{i\in I} f_i \cdot \lambda_i) \geq \delta$ for all $\lambda = (\lambda_i)_{i\in I} \in K^I$ with $v(\lambda) \geq \gamma'$. Set $\gamma_b = \min\{\gamma, \delta\}$. Now if

$$w \in (\sum_{i\in I_1} f_i \cdot K + V_\gamma) \cap (\sum_{i\in I_2} f_i \cdot K + V_\gamma),$$

there is $\lambda = (\lambda_i)_{i\in I} \in K^I$ such that $w \in \sum_{i\in I_1} f_i \cdot \lambda_i + V_\gamma$ and $w \in \sum_{i\in I_2} f_i \cdot \lambda_i + V_\gamma$. Then

$$v(\sum_{i\in I_1} f_i \cdot \lambda_i + \sum_{i\in I_2} f_i \cdot (-\lambda_i)) \geq \gamma,$$

so $v(\lambda) \geq \gamma'$ and therefore $v(\sum_{i\in I_1} f_i \cdot \lambda_i) \geq \delta$. This shows that $w \in V_{\gamma_b}$. $\square$

7.13 Proposition. *Suppose K is maximally valued and M is a finitely generated submodule of N. Then $M \cdot K$ has the weak optimal approximation property modulo large balls.*

Proof. By Proposition 4.12, there exists a strongly independent tuple $(f_i)_{i<n}$ with $f_i \in M$ such that $\{ f_i \mid i < n \}$ generates M. In particular, $M \cdot K = \sum_{i<n} f_i \cdot K$. Since $(f_i)_{i<n}$ is strongly independent, Proposition 6.26 yields $\gamma_0, \delta_0 \in \Gamma$ such that for all $\lambda = (\lambda_i)_{i<n} \in K^n$,

$$\text{(lower)} \qquad \min_{i<n} \phi^{\deg f_i}(v(\lambda_i)) \geq \min\left\{ v(\sum_{i<n} f_i \cdot \lambda_i) + \delta_0, \gamma_0 \right\}.$$

Also, Proposition 6.22 yields $\gamma_1, \delta_1 \in \Gamma$ such that for all $\lambda = (\lambda_i)_{i<n} \in K^n$,

$$\text{(upper)} \qquad v(\sum_{i<n} f_i \cdot \lambda_i) \geq \min\left\{ (\min_{i<n} v(\phi^{\deg f_i}(\lambda_i))) + \delta_1, \gamma_1 \right\}$$

where $\min \emptyset = +\infty$.

Set $\gamma := \min\{\gamma_0, \gamma_1\}$ and $\delta := \max\{-\delta_0 - \delta_1, 0\}$. We will show that $M \cdot K + V_\gamma$ has the δ-optimal approximation property.

We prove by induction on the cardinality of $I \subseteq \{ i \in \mathbb{N} \mid i < n \}$ that $\sum_{i\in I} f_i \cdot K + V_\gamma$ has the δ-optimal approximation property.

For $I = \emptyset$, this is clear. So let $I \subseteq \{ i \in \mathbb{N} \mid i < n \}$ be non-empty, $S = \sum_{i\in I} f_i \cdot K + V_\gamma$, $w \in K^X$ and $\Theta = \{ v(w' - w) \mid w' \in S \}$. For a contradiction, assume that there is no $\theta \in \Theta$ such that Θ is bounded above by $\theta + \delta$ for some $\theta \in \Theta$. Then we can find a limit ordinal $\kappa \geq \omega$ and a strictly increasing sequence $(\theta_j)_{j\in\kappa}$ in Θ that is cofinal in Θ such that $\theta_j + \delta < \theta_{j+1}$ for all $j \in \kappa$. By passing to a subsequence, we may assume that the cofinality of κ is equal to κ, so κ is a cardinal. Note that $\theta_j < \gamma$ for all $j \in \kappa$. In particular, $\Theta = \{ v(w' - w) \mid w' \in \sum_{i\in I} f_i \cdot K \}$. Pick $\mu_{j,i} \in K$ for $j \in \kappa$ and $i \in I$ such that $\theta_j = v((\sum_{i\in I} f_i \cdot \mu_{j,i}) - w)$.

Let $j_1, j_2 \in \kappa$ with $j_1 < j_2$. We have

$$\begin{aligned} v(\sum_{i\in I} f_i \cdot (\mu_{j_1,i} - \mu_{j_2,i})) &= v((\sum_{i\in I} f_i \cdot \mu_{j_1,i} - w) - (\sum_{i\in I} f_i \cdot \mu_{j_2,i} - w)) \\ &= \theta_{j_1} , \end{aligned}$$

because $\theta_{j_1} < \theta_{j_2}$. Set

$$\zeta_{j_1,j_2} := \min_{i\in I} \phi^{\deg f_i}(v(\mu_{j_1,i} - \mu_{j_2,i})) .$$

Note that $\theta_{j_1} + \delta_0 < \gamma_0$, so by the inequality (lower) one obtains

$$\begin{aligned} \zeta_{j_1,j_2} &= \min_{i\in I} \phi^{\deg f_i}(v(\mu_{j_1,i} - \mu_{j_2,i})) \\ &\geq v(\sum_{i\in I} f_i \cdot (\mu_{j_1,i} - \mu_{j_2,i})) + \delta_0 \\ &= \theta_{j_1} + \delta_0 . \end{aligned}$$

Also $\theta_{j_1} < \gamma_1$, so by the inequality (upper) one obtains

$$\begin{aligned} \theta_{j_1} &= v(\sum_{i\in I} f_i \cdot (\mu_{j_1,i} - \mu_{j_2,i})) \\ &\geq (\min_{i\in I} \phi^{\deg f_i}(v(\mu_{j_1,i} - \mu_{j_2,i}))) + \delta_1 \\ &= \zeta_{j_1,j_2} + \delta_1 . \end{aligned}$$

Combining the previous two inequalities, one gets

$$\theta_{j_1} + \delta_0 \leq \zeta_{j_1,j_2} \leq \theta_{j_1} - \delta_1 .$$

Now let $j_1, j_2, j_3 \in \kappa$ with $j_1 < j_2 < j_3$. One has

$$\zeta_{j_1,j_2} \leq \theta_{j_1} - \delta_1 < \theta_{j_2} - \delta - \delta_1 \leq \zeta_{j_2,j_3} - \delta_0 - \delta - \delta_1 \leq \zeta_{j_2,j_3} ,$$

so $\zeta_{j_1,j_2} < \zeta_{j_2,j_3}$ and $\zeta_{j_1,j_2} = \zeta_{j_1,j_3}$. Let $i(j_1, j_2) \in I$ be such that

$$\zeta_{j_1,j_2} = \phi^{\deg f_{i(j_1,j_2)}}(v(\mu_{j_1,i(j_1,j_2)} - \mu_{j_2,i(j_1,j_2)}))\,.$$

Then

$$\phi^{\deg f_{i(j_1,j_2)}}(v(\mu_{j_1,i(j_1,j_2)} - \mu_{j_2,i(j_1,j_2)})) = \phi^{\deg f_{i(j_1,j_2)}}(v(\mu_{j_1,i(j_1,j_2)} - \mu_{j_3,i(j_1,j_2)}))\,.$$

Take $i_{\min} \in I$ and a cofinal $J \subseteq \kappa$ such that $i(j, j+1) = i_{\min}$ for all $j \in J$, hence

$$(*) \qquad \min_{i \in I} \phi^{\deg f_i}(v(\mu_{j_1,i} - \mu_{j_2,i})) = \phi^{\deg f_{i_{\min}}}(v(\mu_{j_1,i_{\min}} - \mu_{j_2,i_{\min}}))$$

for all $j_1, j_2 \in J$. By passing to a subsequence, we may assume that (*) holds for all $j_1, j_2 \in \kappa$. Set $d = \deg f_{i_{\min}}$ and $\mu_j = \mu_{j,i_{\min}}$ for $j \in \kappa$.

For $j_1, j_2, j_3 \in \kappa$ with $j_1 < j_2 < j_3$, we have $v(\mu_{j_1} - \mu_{j_2}) < v(\mu_{j_2} - \mu_{j_3})$, because $\zeta_{j_1,j_2} < \zeta_{j_2,j_3}$ and ϕ is a self-embedding of Γ. Thus $(\mu_j)_{j\in\kappa}$ is a pseudo Cauchy sequence. (For the definition and some facts about pseudo Cauchy sequences, see [Ka].) Since K is maximally valued, we have a pseudo limit $\mu \in K$ for this sequence.

Let $\tilde{I} = I \setminus \{ i_{\min} \}$, $\tilde{w} = w - f_{i_{\min}} \cdot \mu$ and $\tilde{\Theta} = \left\{ v(w' - \tilde{w}) \mid w' \in \sum_{i\in\tilde{I}} f_i \cdot K \right\}$. Clearly $\tilde{\Theta} \subseteq \Theta$.

Claim. The set $\tilde{\Theta}$ is cofinal in Θ.

Let $j_0 \in \kappa$. It suffices to show that $\theta_{j_0} < \tilde{\theta}$ for some $\tilde{\theta} \in \tilde{\Theta}$. Let $l = j_0 + 1$. Then

$$\begin{aligned} v(w - \sum_{i\in I} f_i \cdot \mu_{l,i}) = \theta_l & \\ &= v(\sum_{i\in I} f_i \cdot (\mu_{l,i} - \mu_{l+1,i}) \\ &\le \phi^d(v(\mu_l - \mu_{l+1})) - \delta_0 \\ &\le \phi^d(v(\mu_l - \mu)) - \delta_0\,, \end{aligned}$$

because $v(\mu_l - \mu) = v(\mu_l - \mu_{l+1})$, since μ is a pseudo limit of $(\mu_j)_{j\in\kappa}$. Thus

$$\begin{aligned} v((\sum_{i\in\tilde{I}} f_i \cdot \mu_{l,i}) - \tilde{w}) &= v((\sum_{i\in I} f_i \cdot \mu_{l,i}) - w + f_{i_{\min}} \cdot (\mu - \mu_l)) \\ &\ge \min \left\{ v((\sum_{i\in I} f_i \cdot \mu_{l,i}) - w), v(f_{i_{\min}} \cdot (\mu - \mu_l)) \right\} \\ &= \min \{ \theta_l, v(f_{i_{\min}} \cdot (\mu_l - \mu)) \} \\ &\ge \min \left\{ \theta_l, v(\phi^d(\mu_l - \mu)) + \delta_1, \gamma_1 \right\} \quad \text{by inequality (upper)} \\ &\ge \min \{ \theta_l, \theta_l + \delta_0 + \delta_1 \} \\ &> \min \{ \theta_{j_0} + \delta, \theta_{j_0} + \delta + \delta_0 + \delta_1 \} \\ &\ge \theta_{j_0} \end{aligned}$$

and since $v((\sum_{i\in\tilde I} f_i\cdot\mu_{l,i})-\tilde w)\in\tilde\Theta$, the claim is proved.

By induction assumption, $\hat S := \sum_{i\in\tilde I} f_i\cdot K + V_\gamma$ has the δ-optimal approximation property, so there exist $\theta\in\hat\Theta := \left\{\, v(w'-\tilde w)\mid w'\in\hat S\,\right\}$ such that $\theta+\delta$ is an upper bound for $\hat\Theta$. We have $\tilde\Theta\subseteq\hat\Theta\subseteq\Theta$, so $\theta+\delta$ is an upper bound for Θ by the claim, and we arrive at a contradiction. $\square$

7.14 Lemma. *Let M_1 and M_2 be finitely generated submodules of N such that $M_1\subseteq M_2$.*

1.
$$M_1\cdot K \overset{\infty}{=} M_2\cdot K \implies \dim_\infty M_1 = \dim_\infty M_2\,.$$

2. *Suppose that for every finitely generated submodule M of N, the set $M\cdot K$ has the weak optimal approximation property modulo large balls. (Hence by Proposition 7.5 every pp-definable subset of K^X has the weak optimal approximation property modulo large balls.) Then*
$$\dim_\infty M_1 = \dim_\infty M_2 \implies M_1\cdot K \overset{\infty}{=} M_2\cdot K\,.$$

Proof. Because $M_1\subseteq M_2$, one has $\dim_\infty M_1\le\dim_\infty M_2$ and $M_1\cdot K\subseteq M_2\cdot K$.

To prove the first statement, assume that $\dim_\infty M_1<\dim_\infty M_2$. We will show that $M_2\cdot K\subseteq M_1\cdot K+B$ holds for no ball B in K^X. Pick a strongly independent finite tuple $(f_i)_{i\in I}$ of elements of N such that $\{\, f_i\mid i\in I\,\}$ generates M_1. Let $d\in\mathbb{N}$ be such that $d\ge\deg f_i$ for all $i\in I$ and d is an upper bound for the degrees of the elements of some generating set of M_2. Because $\dim_\infty M_1<\dim_\infty M_2$ and $M_1\subseteq M_2$, we know that $(M_1\cap N_{\le d})+N_{<d}$ is properly contained in $(M_2\cap N_{\le d})+N_{<d}$ by Proposition 4.8. Pick $g\in(M_2\cap N_{\le d})\setminus((M_1\cap N_{\le d})+N_{<d})$. Then by Remark 4.11, part 5, the tuple $(f_i)_{i\in I}{}^\frown(g)$ is strongly independent. Given a ball B in K^X, there exists by Lemma 7.12 a ball B' in K^X such that
$$(\sum_{i\in I} f_i\cdot K+B)\cap(g\cdot K+B)\subseteq B'\,.$$
Because Γ is not zero, the set $g\cdot K$ is unbounded and therefore not contained in $\sum_{i\in I} f_i\cdot K+B = M_1\cdot K+B$.

For the proof of the second statement, assume that $\dim_\infty M_1=\dim_\infty M_2$. Pick $d\in\mathbb{N}$ such that d is an upper bound for the degrees of the elements of some generating set of M_1 and also an upper bound for the degrees of the elements of some generating set of M_2. Let $M_h := M_1\cap N_{\le d}$. Because $\dim_\infty M_1=\dim_\infty M_2$, we have $M_2\cap N_{\le d}\subseteq M_h+N_{<d}$. Let $M_l=M_2\cap N_{<d}$. Note that $M_2\cdot K=M_h\cdot K+M_l\cdot K$. By Lemma 7.11 and the implication $2\implies 3$ of Lemma 7.8, $M_2\cdot K\overset{\infty}{\subseteq} M_h\cdot K$; thus $M_2\cdot K\overset{\infty}{\subseteq} M_1\cdot K$. $\square$

7.15 Assumption. For the rest of the section, the set $M\cdot K\subseteq K$ has the weak optimal approximation property modulo large balls for every finitely generated submodule M of $K[\Phi]$.

Then for every finite set X and for every finitely generated submodule M of $K[\Phi]^X$, the set $M\cdot K$ has the weak optimal approximation property modulo large balls by Lemma 7.8, $1\implies 2$.

7.16 Definition. Suppose S is a pp-definable subset of K^X. By Proposition 7.5, there exists a finitely generated submodule M of $N = K[\Phi]^X$ such that $S \overset{\infty}{\cong} M \cdot K$. By Lemma 7.14, the value $\dim_\infty M$ does not depend on the choice of M. Define $\dim_\infty S$ as $\dim_\infty M$.

7.17 Theorem. *Suppose S, S_1, S_2 are pp-definable subsets of K^X. Then*

1. $S_1 \overset{\infty}{\subseteq} S_2 \implies \dim_\infty S_1 \leq \dim_\infty S_2$,
2. $(S_1 \overset{\infty}{\subseteq} S_2$ *and* $\dim_\infty S_1 = \dim_\infty S_2) \implies S_1 \overset{\infty}{\cong} S_2$,
3. $\dim_\infty \{ 0 \} = 0$,
4. $\dim_\infty K^X = |X|$.

Proof. For the first two statements, assume $S_1 \overset{\infty}{\subseteq} S_2$ and pick for $i = 1, 2$ finitely generated submodules M_i of N with $M_i \cdot K \overset{\infty}{\cong} S_i$ using Proposition 7.5. Let $M_2' = M_1 + M_2$. Then $M_2' \cdot K \overset{\infty}{\cong} M_2 \cdot K$, so replacing M_2 by M_2' we may assume that $M_1 \subseteq M_2$. The first two statements are now direct consequences of Lemma 7.14.

For the submodule $M = \{ 0 \}$ of N, we have $\dim_\infty M = 0$ and $M \cdot K = \{ 0 \}$, so $\dim_\infty \{ 0 \} = 0$. For the submodule $M = N$, we have $\dim_\infty M = |X|$ and $M \cdot K = K^X$, so $\dim_\infty K^X = |X|$. □

7.18 Remark. 1. The quantity $\dim_\infty$ for pp-definable sets is not always invariant under definable bijections, not even when the definable bijection on K^X is induced by multiplying with an elementary matrix in $\mathrm{MAT}_X(K[\Phi])$: Suppose that $t \in K \setminus \phi(K)$, $X = \{ 1, 2 \}$,

$$D = \begin{pmatrix} t\Phi & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad E = \begin{pmatrix} 1 & \Phi \\ 0 & 1 \end{pmatrix}$$

and M is the submodule of N generated by the columns of D and M' is the submodule of N generated by the columns of ED. Then

$$\dim_\infty M = 1 + [K : \phi(K)]^{-1} \text{ and } \dim_\infty M' = 2[K : \phi(K)]^{-1}$$

by proposition 4.12, part 3, since

$$ED = \begin{pmatrix} t\Phi & \Phi \\ 0 & 1 \end{pmatrix}$$

and the columns of both D and ED are strongly independent. So with $S := M \cdot K$ and $S' := M' \cdot K$, we have $\dim_\infty S \neq \dim_\infty S'$, but $\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \mapsto E \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ is a bijection $K^X \longrightarrow K^X$ that sends S onto S'.

2. For pp-definable sets $S_1, S_2 \subseteq K^X$, one does not have in general that $\dim_\infty S_1 + S_2 \leq \dim_\infty S_1 + \dim_\infty S_2$, even if $\dim_\infty S_1 \cap S_2 = 0$: Assume that $t \in K \setminus \phi(K)$ and consider $S_1 = \Phi^2 \cdot K$ and $S_2 = (\Phi^2 - t\Phi) \cdot K$. Then $S_1 + S_2 = \Phi^2 \cdot K + t\Phi \cdot K$, and hence

$$\dim_\infty S_1 + S_2 = [K : \phi(K)]^{-2} + [K : \phi(K)]^{-1} ,$$

because Φ^2 and $t\Phi$ are strongly independent. On the other hand, $\dim_\infty S_i = [K : \phi(K)]^{-2}$ for $i = 1, 2$. Also $S_1 \cap S_2$ is bounded by Example 7.6, so $\dim_\infty S_1 \cap S_2 = 0$.

7.19 Proposition. *Suppose there exists a pp-definable set $C \subseteq K$ that is bounded and contains a ball. Let S be a pp-definable set in K^X. Then there exists a pp-definable set S_c in K^X such that $S + S_c = K^X$ and $S \cap S_c$ is bounded.*

Proof. Take a finitely generated submodule M of N such that $S \overset{\infty}{=} M \cdot K$. Let $(f_j)_{j\in J}$ be a strongly independent finite tuple of elements of N such that $\{ f_j \mid j \in J \}$ generates M. Let $V = \sum_{j\in J} G(f_j) \subseteq K^X$. Let $d \in \mathbb{N}$ such that $d \geq \deg f_j$ for all $j \in J$. Then V is a $\phi^d(K)$-subspace of K^X. Pick a set J' disjoint from J and $v_j \in K^X$ for $j \in J'$ such that $(v_j)_{j\in J'}$ is a $\phi^d(K)$-basis of a $\phi^d(K)$-subspace of K^X that is complementary to V in K^X.

For $j \in J'$, let $f_j := v_j\Phi^d$. Then the tuple $(f_j)_{j\in J\cup J'}$ is strongly independent and $\sum_{j\in J\cup J'} G(f_j) = K^X$. Let M' be the submodule of N generated by $\{ f_j \mid j \in J' \}$. We have $\dim_\infty M \cdot K + M' \cdot K = \dim_\infty M + M' = |X|$, thus $M \cdot K + M' \cdot K \overset{\infty}{=} K^X$. Since C contains a ball, there exists $\lambda \in K$ such that $M \cdot K + M' \cdot K + (\lambda C)^X = K^X$ and $M \cdot K + (\lambda C)^X = S + (\lambda C)^X$. Let $S_c = M' \cdot K + (\lambda C)^X$; then $S + S_c = K^X$. The set S_c is clearly pp-definable. Because $(\lambda C)^X$ is bounded and $(f_j)_{j\in J\cup J'}$ strongly independent, Lemma 7.12 yields that $S \cap S_c$ is bounded. ☐

7.2 The small case ($v(x) \to +\infty$)

In this section, we look at the structure of pp-definable sets near 0, i.e. after intersecting with an appropriately small ball. Let B be a basis of K over $\phi(K)$, and let X and Y denote finite index sets. We assume in this section that for each $P \in \mathcal{P}$ either $P = \{ 0 \}$ or P contains a ball. In this section, all $K[\Phi]^J$ for finite J are considered as left $K[\Phi]$-modules, as in Section 4.2.

7.20 Definition.

1. For subsets $S_1, S_2 \subseteq K^X$, say that S_1 is **contained** in S_2 **inside small balls**, if $S_1 \cap U \subseteq S_2 \cap U$ for some ball U in K^X, and denote this by $S_1 \overset{0}{\subseteq} S_2$. Similarly, call S_1 and S_2 **equal inside small balls**, if $S_1 \cap U = S_2 \cap U$ for some ball U in K^X, and denote this by $S_1 \overset{0}{=} S_2$.

2. We call the valued field K **linear ϕ-henselian**, if for every $f \in K[\Phi] \setminus K[\Phi]\Phi$ and every ball U in K, there exists a ball V in K such that $f \cdot U \supseteq V$.

7.21 Remark. If K is of characteristic p and henselian and ϕ is the Frobenius self-embedding $\lambda \mapsto \lambda^p$, then K is linear ϕ-henselian.

We equip K^X with the product topology of the valuation topology on K. The translates of balls in K^X form a base of this topology on K^X.

7.22 Lemma. *Suppose K is maximally valued. Then K is linear ϕ-henselian.*

Proof. Let $f = \sum_{i=0}^{n} \mu_i \Phi^i \in K[\Phi] \setminus K[\Phi]\Phi$. We want to show that for every ball U in K, there exists a ball V in K such that $f \cdot U \supseteq V$. Because the map $K \longrightarrow K$, $\lambda \mapsto \mu_0^{-1}\lambda$ is a homeomorphism of K, we may assume that $\mu_0 = 1$. Let $h : K \longrightarrow K$ be the map given by $h(\lambda) = (\sum_{i=1}^{n} \mu_i \Phi^i) \cdot \lambda$.

Claim. Given $\delta \in \Gamma$ with $\delta \geq 0$, there exists $\gamma_0 \in \Gamma$ such that $h(V_\gamma) \subseteq V_{\gamma+\delta}$ for all $\gamma \geq \gamma_0$.

If $\sum_{i=1}^{n} \mu_i \Phi^i = 0$, the claim is clear. Suppose $\sum_{i=1}^{n} \mu_i \Phi^i \neq 0$. Then by Proposition 6.20, there exist $\gamma_+, \delta_0 \in \Gamma$ and $d \in \mathbb{N}$ with $d > 0$ such that $v(h(\lambda)) = \phi^d(v(\lambda)) + \delta_0$ for all $\lambda \in K$ with $v(\lambda) > \gamma_+$. Let $\delta \in \Gamma$. We can apply Lemma 6.14 to obtain $\gamma'_+ \in \Gamma$ such that $\gamma + \delta \leq \phi^d(\gamma) + \delta_0$ for all $\gamma \geq \gamma'_+$. Pick $\gamma_0 > \gamma_+, \gamma'_+$ using the assumption that Γ is non-trivial. Then we have $h(V_\gamma) \subseteq V_{\gamma+\delta}$ for all $\gamma \geq \gamma_0$, so the claim is proved.

Now let U be a ball in K. Pick any $\delta > 0$ in Γ and apply the claim to obtain $\gamma_0 \in \Gamma$ such that $V_{\gamma_0} \subseteq U$ and $h(V_\gamma) \subseteq V_{\gamma+\delta}$ for all $\gamma \geq \gamma_0$. For any $a \in V_{\gamma_0}$, the map $h_a : K \longrightarrow K, \lambda \mapsto a - h(\lambda)$ restricts to a map from V_{γ_0} into V_{γ_0}, and this restriction is δ-contractive. Therefore h_a has a fixed point b in V_{γ_0} by Lemma 6.31. So $b = a - h(b)$, and therefore $f \cdot b = a$. This shows that $f \cdot V_{\gamma_0} \supseteq V_{\gamma_0}$, so we can choose $V = V_{\gamma_0}$. □

7.23 Lemma.

1. *Suppose that* $M \in K[\Phi]^{Y \times X}$. *Then the map* $K^X \longrightarrow K^Y$, $w \mapsto M \cdot w$ *is continuous.*

2. *The bijection* $K^B \longrightarrow K$ *given by* $(\lambda_b)_{b \in B} \mapsto \sum_{b \in B} \phi(\lambda_b) b$ *is a homeomorphism.*

Proof. Part 1 follows from Lemma 6.20, part 1. Part 2 follows from Lemma 6.25. □

7.24 Lemma. *Suppose that* $M \in K[\Phi]^{Y \times X}$ *is column regular. Then there exists a ball* U *in* K^X *such that the map* $K^X \longrightarrow K^Y$, $w \mapsto M \cdot w$ *restricted to* U *is an injection.*

Proof. By Lemma 3.9, we may assume that M is in upper triangular form with respect to $\leq$ and ι. We may assume that the upper triangular form has no zero part. Because M is column regular, ι is a bijection, and $\operatorname{ldeg} M(y, \iota(y)) = 0$ for $y \in Y$.

Use Lemma 6.20, part 1 to find a ball W in K such that

$$M(y, \iota(y)) \cdot \lambda = 0 \implies \lambda = 0$$

for $y \in Y$ and $\lambda \in W$. Then the map $w \mapsto M \cdot w$ is an injection on $U = W^X$. □

7.25 Lemma. *Suppose that* $M \in K[\Phi]^{Y \times X}$. *If there exists a ball* U *in* K^X *such that* $M \cdot U = \{0\} \subseteq K^Y$, *then* $M = 0$.

Proof. It suffices to show this in the case where $|X| = 1$ and $|Y| = 1$. Apply Lemma 6.20, part 1. □

7.26 Lemma. *Assume that* K *is linear* ϕ*-henselian. Then the following generalization of the linear* ϕ*-henselian property holds for matrices: Let* $M \in K[\Phi]^{Y \times X}$ *be row regular. Then for every ball* U *in* K^X, *there exist a ball* V *in* K^Y *such that* $M \cdot U \supseteq V$.

Proof. It suffices to prove the conclusion of the lemma with M replaced by EM where $E \in \mathrm{GL}_Y(K[\Phi])$. So by Lemma 3.9, we can assume M is in upper triangular form (with empty zero part) with respect to the injection $\iota : X \longrightarrow Y$ and the total order $\leq$ on X.

Let U be a ball in K^X. We apply the henselian property to the diagonal elements inductively using continuity in the other elements:

Claim. Let U_0 be a ball in K such that $U_0^X \subseteq U$. There exist balls V_y, W_y in K for $y \in Y$ such that for all $y \in Y$, we have

1. $V_y \subseteq U_0$,
2. $M(y', \iota(y)) \cdot V_y \subseteq W_{y'}$ for all $y' \in Y$ with $y' < y$,
3. $M(y, \iota(y)) \cdot V_y \supseteq W_y$.

Denote the conjunction of the 3 Properties by $P(y)$. Let $y \in Y$, and suppose V'_y, W'_y are balls in K for $y' < y$ such that for each $y' < y$ we have $P(y')$. Since the map $K \longrightarrow K$, $\lambda \mapsto M(y', \iota(y)) \cdot \lambda$ is continuous for $y' \in Y$, we can pick V_y that satisfies the first two properties. Since M is row regular, we have $M(y, \iota(y)) \notin K[\Phi]\Phi$, so the existence of W_y satisfying the third property follows, because K is linear ϕ-henselian. This finishes the proof of the claim.

It follows from the claim that

$$M \cdot U \supseteq M\restriction_{Y \times \iota(Y)} \cdot \prod_{x \in \iota(Y)} V_{\iota^{-1}(x)} \supseteq \prod_{y \in Y} W_y \,.$$

□

7.27 Proposition. *Assume that K is linear ϕ-henselian. Let $A \subseteq K^X$ be a pp-definable subgroup. Then there exist $c \in \mathbb{N}$ and a separable submodule N of $K[\Phi]^J$ with $J := X \times B^c$ such that $T^c(A) \overset{0}{=} \mathrm{Ann}(N)$ where T^c is the canonical bijection $K^X \longrightarrow K^{X \times B^c} = K^J$ determined by the basis B, as defined in Remark 3.11.*

This means that inside some ball in K^J, the image of A under T^c is the same as the solution set of a finite system of linear equations. So to study the small asymptotic behaviour of pp-definable sets, we can reduce to the simpler situation of solution sets of linear equations.

Proof. Since A is pp-definable, we can find finite index sets H and Y with Y disjoint from X, $S \in K[\Phi]^{H \times (X \cup Y)}$ and $P \in \mathcal{P}^H$ such that A is defined by the formula $\exists Y\, (S \cdot (X \cup Y) \in P)$. Let $I := \{\, i \in H \mid P_i = \{\, 0 \,\}\,\}$ and

$$H^* := \{\, i \in H \mid P_i \text{ contains a ball in } K \,\}\,.$$

Then $I \cap H^* = \emptyset$ and $I \cup H^* = H$. Pick a ball U in K such that $U \subseteq P_i$ for all $i \in H^*$.

Apply Lemma 3.15 to the matrix $M := S\restriction_{I\times(X\cup Y)}$ to obtain $c \in \mathbb{N}$, disjoint finite index sets $\hat{I}_1, \hat{I}_2$, and $\hat{M} \in K[\Phi]^{\hat{I}\times\hat{J}}$, where $\hat{J} = J\dot{\cup}Y$ and $\hat{I} = \hat{I}_1 \cup \hat{I}_2$, such that with

$$\begin{aligned}
\hat{M}_{11} &:= \hat{M}\restriction_{\hat{I}_1\times J} \in K[\Phi]^{\hat{I}_1\times J}\,,\\
\hat{M}_{12} &:= \hat{M}\restriction_{\hat{I}_1\times Y} \in K[\Phi]^{\hat{I}_1\times Y}\,,\\
\hat{M}_{21} &:= \hat{M}\restriction_{\hat{I}_2\times J} \in K[\Phi]^{\hat{I}_2\times J} \text{ and}\\
\hat{M}_{22} &:= \hat{M}\restriction_{\hat{I}_2\times Y} \in K[\Phi]^{\hat{I}_2\times Y}\,,
\end{aligned}$$

we have the following properties:

1. $\hat{M}_{12}$ is row regular.
2. $\hat{M}_{22} = 0$.
3. The non-zero rows of $\hat{M}_{21}$ form a row regular matrix.
4. For all $u \in K^X$ and $w \in K^Y$, we have

$$M \cdot (u^\frown w) = 0 \iff \hat{M} \cdot (T^c(u)^\frown w) = 0\,.$$

Note that $\hat{M}$ has the form $\left(\begin{smallmatrix}\hat{M}_{11}\hat{M}_{12}\\ \hat{M}_{21}\hat{M}_{22}\end{smallmatrix}\right)$.

Let N be the submodule of $K[\Phi]^J$ generated by the rows of $\hat{M}_{21}$. This submodule is separable because of the condition 3 on $\hat{M}$ above. In the following, we will construct a ball W in K^J such that $T^c(A) \cap W = \mathrm{Ann}(N) \cap W$.

First pick a ball U_X in K^X and a ball U_Y in K^Y such that $S\cdot(U_X \times U_Y) \subseteq U^H$. By Lemma 7.26, there exists a ball V in $K^{\hat{I}_1}$ with $V \subseteq \hat{M}_{12} \cdot U_Y$, since $\hat{M}_{12}$ is row regular and K linear ϕ-henselian. Choose W as a ball in K^J such that $\hat{M}_{11} \cdot W \subseteq V$ and $W \subseteq T^c(U_X)$.

To show $T^c(A) \cap W \subseteq \mathrm{Ann}(N) \cap W$, let $u \in A$ such that $\hat{u} := T^c(u) \in W$. Then there exists $w \in U^Y$ such that $S\cdot(u^\frown w) \in \prod_{i\in H} P_i$. In particular, $M\cdot(u^\frown w) \in \prod_{i\in I} P_i = \{0\}^I$, so $M\cdot(u^\frown w) = 0$. By property 4 on $\hat{M}$, it follows that $\hat{M} \cdot (\hat{u}^\frown w) = 0$. In particular, $0 = \hat{M}_{21} \cdot \hat{u} + \hat{M}_{22} \cdot w = \hat{M}_{21} \cdot \hat{u}$ and thus $\hat{u} \in \mathrm{Ann}(N)$.

To show the other inclusion, let $\hat{u} \in \mathrm{Ann}(N) \cap W$. So by definition of N, we have $\hat{M}_{21} \cdot \hat{u} = 0$. Since $\hat{M}_{11} \cdot W \subseteq V$ and thus $-\hat{M}_{11} \cdot \hat{u} \in V$, we can find $w \in U_Y$ with $\hat{M}_{12} \cdot w = -\hat{M}_{11} \cdot \hat{u}$. We get $\hat{M} \cdot (\hat{u}^\frown w) = 0$ and therefore by condition 4 on $\hat{M}$ above, $M \cdot (u^\frown w) = 0 \in \prod_{i\in I} P_i$ where $u \in K^X$ satisfies $(T^c)(u) = \hat{u}$. Since $W \subseteq T^c(U_X)$, we know that $S\cdot(u^\frown w) \in U^H$, so with $U^{H^*} \subseteq \prod_{h\in H^*} P_h$, we obtain $K \models S \cdot (u^\frown w) \in P$. ☐

7.28 Corollary. *Assume that K is linear ϕ-henselian. Then each pp-definable subgroup of K^X is closed; if $\Gamma = \mathbb{Z}$, then each pp-definable subgroup of K^X has the optimal approximation property.*

This corollary is a generalization of theorem 1 in [VDK].

Proof. Let A be a pp-definable subgroup of K^X, and take c and N as in Proposition 7.27. Since $T^c : K^X \longrightarrow K^J$ is a homeomorphism and an additive group homomorphism, it suffices to show that $T^c(A)$ is closed in some ball in K^J, which holds, because $T^c(A) \overset{0}{=} \mathrm{Ann}(N)$ and $\mathrm{Ann}(N)$ is closed. □

7.29 Remark. Let $M \subseteq K[\Phi]^X$ be a submodule. Then $\mathrm{Ann}(\mathrm{col}(M)) \overset{0}{=} T(\mathrm{Ann}(M))$ where T is the canonical bijection $K^X \longrightarrow K^X = K^{X\times B}$ determined by the basis B and col is the homomorphism in Remark 4.23.

In particular, the conclusion of the previous proposition can be strengthened in the following way: There exists $c' \in \mathbb{N}$ such that for every $c \in \mathbb{N}$ with $c \geq c'$ there exists a submodule N of $K[\Phi]^J$ with $J := X \times B^c$ such that $T^c(A) \overset{0}{=} \mathrm{Ann}(N)$.

7.30 Lemma. *Let J be a finite set and M_1, M_2 be submodules of $K[\Phi]^J$ such that $M_1 \subseteq M_2$.*

1.
$$\dim_0 M_1 = \dim_0 M_2 \implies \mathrm{Ann}(M_1) \overset{0}{=} \mathrm{Ann}(M_2)\,.$$

2. If K is linear ϕ-henselian, then
$$\mathrm{Ann}(M_1) \overset{0}{=} \mathrm{Ann}(M_2) \implies \dim_0 M_1 = \dim_0 M_2\,.$$

Proof. If M is a submodule of $K[\Phi]^J$ and M' is the separable submodule generated by M, then by Remark 5.4, part 2, $\mathrm{Ann}(M) = \mathrm{Ann}(M')$ and by definition, $\dim_0 M = \dim_0 M'$. So without loss of generality, we may assume that M_1 and M_2 are separable.

For the proof of the first statement, assume that $\dim_0 M_1 = \dim_0 M_2$. By Lemma 4.28, we have $M_2 \subseteq (M_1)_{\mathfrak{m}}$. Let C be a finite generating set of M_2. For each $f \in C$, pick $g_f \in M_1$ and $h_f \in K[\Phi] \setminus \mathfrak{m}$ with $f = h_f^{-1} g_f$, so $h_f f = g_f$. Use Lemma 7.24 to pick a ball U in K such that for all $\lambda \in U$ and $f \in C$, we have
$$h_f \cdot \lambda = 0 \implies \lambda = 0\,.$$
Then pick a ball V in K^J such that $f \cdot V \subseteq U$ for $f \in C$.

We will show that $\mathrm{Ann}(M_1) \cap V \subseteq \mathrm{Ann}(M_2) \cap V$; the other inclusion is obvious. Let $w \in \mathrm{Ann}(M_1) \cap V$ and $f \in C$. Then $g_f \cdot w = 0$, so $0 = (h_f f) \cdot w = h_f \cdot (f \cdot w)$. Since $f \cdot w \in U$, we get $f \cdot w = 0$. This shows that $w \in \mathrm{Ann}(C) = \mathrm{Ann}(M_2)$.

We now give the proof of the second statement. Assume that K is linear ϕ-henselian and $\mathrm{Ann}(M_1) \cap U = \mathrm{Ann}(M_2) \cap U$ where U is a ball in K^J. By Lemma 4.28, it suffices to show $M_2 \subseteq (M_1)_{\mathfrak{m}}$. Since M_1 is separable, there exist $J_0 \subseteq J$ and a row regular matrix $M \in K[\Phi]^{J_0 \times J}$ that is in upper triangular form with respect to J_0 and $\iota = \mathrm{id}_{J_0}$ such that the rows of M generate M_1.

Claim. There exists a ball W in $K^{J\setminus J_0}$ such that $W \subseteq \pi_{J\setminus J_0}(\mathrm{Ann}(M_1) \cap U)$ where $\pi_{J\setminus J_0} : K^J \longrightarrow K^{J\setminus J_0}$ denotes the canonical projection.

To find W, first pick balls U_0 in K^{J_0} and U_1 in $K^{J\setminus J_0}$ such that $U_0 \times U_1 \subseteq U$. Using that K is linear ϕ-henselian, by Lemma 7.26, we take a ball V in K^{J_0} such that $V \subseteq (M\restriction_{J_0\times J_0}) \cdot U_0$. Then choose a ball $W \subseteq U_1$ in $K^{J\setminus J_0}$ such that $(M\restriction_{J_0\times(J\setminus J_0)}) \cdot W \subseteq V$. Given $w \in W$, we can find $u_0 \in U_0$ such that $(M\restriction_{J_0\times J_0}) \cdot u_0 = -(M\restriction_{J_0\times(J\setminus J_0)}) \cdot w$, so $M \cdot (u_0 ^\frown w) = 0$. The claim is proved.

Suppose that $f \in M_2$. Using that the diagonal elements of M are invertible in $K[\Phi]_{\mathfrak{m}}$, one can find an element $g \in (M_1)_{\mathfrak{m}}$ such that for $f' = f - g$ we have $f'_j = 0$ for $j \in J_0$. Write $f' = h^{-1}f''$ with $h \in K[\Phi] \setminus \mathfrak{m}$ and $f'' \in M_2$. It suffices to show $f''_j = 0$ for $j \in J \setminus J_0$. By the claim and $\mathrm{Ann}(M_1) \cap U = \mathrm{Ann}(M_2) \cap U$, we see that $f''\restriction_{J\setminus J_0} \cdot W = \{0\}$ and thus $f''\restriction_{J\setminus J_0} = 0$ by Lemma 7.25. □

7.31 Assumption. For the rest of the section, K is linear ϕ-henselian.

7.32 Lemma/Definition. Let $A \subseteq K^X$ be a pp-definable subgroup.

1. There exist $c \in \mathbb{N}$ and a separable submodule M of $K[\Phi]^J$ with $J := X \times B^c$ such that $T^c_B(A) \overset{0}{=} \mathrm{Ann}(M)$ where T^c_B is the canonical bijection $K^X \longrightarrow K^{X\times B^c} = K^J$ determined by the basis B.

2. Let c and M be as in part 1. Let B' be a basis of K over $\phi(K)$ and $c' \in \mathbb{N}$. Let M' be a separable submodule of $K[\Phi]^{J'}$ with $J' := X \times B'^{c'}$ such that $T^{c'}_{B'}(A) \overset{0}{=} \mathrm{Ann}(M')$ where $T^{c'}_{B'}$ is the canonical bijection $K^X \longrightarrow K^{X\times B'^{c'}} = K^{J'}$ determined by the basis B'. Then

$$|X| - \frac{\dim_0 M}{|B|^c} = |X| - \frac{\dim_0 M'}{|B'|^{c'}}.$$

   Define $\dim_0 A$ as this common quantity.

Proof. Part 1 is Proposition 7.27. For part 2, assume without loss of generality that $c \leq c'$. Let $\mathrm{col}(M)$ denote the column enlargement of M with respect to B, which is defined in Remark 4.23. By using Lemma 4.24 and $\mathrm{Ann}(\mathrm{col}(M)) \overset{0}{=} T_B(\mathrm{Ann}(M))$, we may assume $c = c'$. There is some invertible matrix $E \in K[\Phi]^{B^c\times B'^c}$ such that for all $a \in K^{B^c}$, we have $T^c_{B'}(a) = E \cdot T^c_B(a)$. In the case where $c = 1$, the matrix E can be chosen as the unique $E \in \phi(K)^{B\times B'}$ such that $b = \sum_{b'\in B'} E(b,b')b'$ for all $b \in B$. Let $\hat{E} \in K[\Phi]^{(X\times B^c)\times(X\times B'^c)}$ be given by

$$\hat{E}((x,b),(x',b')) = \begin{cases} E(b,b') & \text{for } x = x' \\ 0 & \text{for } x \neq x' \end{cases}$$

for $x, x' \in X$, $b \in B$ and $b' \in B'$. Then $\hat{E}$ is invertible. We have $\hat{E} \cdot \mathrm{Ann}(M) \overset{0}{=} \mathrm{Ann}(M')$ and $\hat{E} \cdot \mathrm{Ann}(M) = \mathrm{Ann}(M\hat{E}^{-1})$, so $\dim_0 M\hat{E}^{-1} = \dim_0 M'$ by Lemma 7.30. Right multiplication by $\hat{E}^{-1}$ induces a module isomorphism from $K[\Phi]^{X\times B^c}$ onto $K[\Phi]^{X\times B'^c}$. So by Remark 4.20, part 3, we have $\dim_0 M\hat{E}^{-1} = \dim_0 M$. □

7.33 Theorem. *Suppose S, S_1, S_2 are pp-definable subsets of K^X. Then:*

1. $S_1 \overset{0}{\subseteq} S_2 \implies \dim_0 S_1 \leq \dim_0 S_2$.

2. $(S_1 \overset{0}{\subseteq} S_2$ *and* $\dim_0 S_1 = \dim_0 S_2) \implies S_1 \overset{0}{=} S_2$.

3. $\dim_0 \{ 0 \} = 0$.

4. $\dim_0 K^X = |X|$.

Proof. For the first two statements, assume $S_1 \overset{0}{\subseteq} S_2$. By Proposition 7.27 and Remark 7.29, one can find $c \in \mathbb{N}$ and submodules N_1, N_2 of $K[\Phi]^J$ with $J := X \times B^c$ such that $T^c(S_i) \overset{0}{=} \mathrm{Ann}(N_i)$ for $i = 1, 2$. Let $M_1 = N_1 + N_2$ and $M_2 = N_2$. Then $T^c(S_1) \overset{0}{=} \mathrm{Ann}(M_1)$ because $S_1 \overset{0}{\subseteq} S_2$. The first two statements follow now from Lemma 7.30.

For the submodule $M = K[\Phi]^X$, we have $\dim_0 M = |X|$ and $\mathrm{Ann}(M) = \{ 0 \}$, so $\dim_0 \{ 0 \} = |X| - |X| = 0$. For the submodule $M = \{ 0 \}$ of $K[\Phi]^X$, we have $\dim_0 M = 0$ and $\mathrm{Ann}(M) = K^X$, so $\dim_0 K^X = |X| - 0 = |X|$. □

7.34 Definition. Suppose $h : K^X \longrightarrow K^Y$ is an additive group homomorphism. We say that h has the **ϕ-linear Greenberg property** (compare [G], p.563, theorem 1), if for every ball V in K^X, there exists a ball U in K^Y such that for all $a \in K^X$, one has

$$h(a) \in U \implies \exists a' \in K^X \, (a - a' \in V \wedge h(a') = 0) \, .$$

7.35 Proposition (ϕ-linear Greenberg theorem). *Suppose $M \in K[\Phi]^{Y \times X}$. Then the map $h : K^X \longrightarrow K^Y, a \mapsto M \cdot a$ has the ϕ-linear Greenberg property.*

Proof. Suppose Y' is a finite set and $h' : K^Y \longrightarrow K^{Y'}$ is an additive group isomorphism that is also a homeomorphism. Then it suffices to show that $h' \circ h$ has the ϕ-linear Greenberg property. So by Lemma 3.14, we may assume that M is in upper triangular form with row regular non-zero part. Then it suffices to show the ϕ-linear Greenberg property for the map given by the non-zero part of M, so we may assume that M is row regular. Let V be a ball in K^X. By Lemma 7.26, there exists a ball U in K^Y such that $M \cdot V \supseteq U$. Let $a \in K^X$ be such that $M \cdot a \in U$. Then there exists $b \in V$ such that $M \cdot b = M \cdot a$, so $a' = a - b$ satisfies $a - a' \in V$ and $M \cdot a' = 0$. □

Chapter 8

Reduction to Factor Modules and Residue Field

In this chapter, let (K, v, Γ) be a valued field (with $v(K^\times) = \Gamma \neq \{ 0 \}$) and ϕ be a self-embedding of the valued field K that has a modulus of growth. We assume

1. that K has a weakly valuation independent basis over $\phi(K)$ and $[K : \phi(K)]$ is finite,
2. that for every finite set X and finitely generated submodule M of $K[\Phi]^X$, the set $M \cdot K$ has the weak optimal approximation property modulo large balls, and
3. that K is linear ϕ-henselian.

In addition, we fix a set $\mathcal{P}$ of unary predicate symbols such that $V_\infty \in \mathcal{P}$. Every $P \in \mathcal{P}$ is interpreted as an additive subgroup of K. This interpretation is also denoted by P and V_∞ is interpreted as the zero subgroup. We assume that for every $P \in \mathcal{P}$, the subset P of K is bounded and either contains a ball in K or is equal to $\{ 0 \}$.

Let $V = V_0 = \{ \lambda \in K \mid v(\lambda) \geq 0 \}$ be the valuation ring of K, $V_{>0} = \{ \lambda \in K \mid v(\lambda) > 0 \}$ its maximal ideal, and $\mathrm{k} = V/V_{>0}$ the residue field of K. Note that $\phi(V) \subseteq V$, so ϕ induces a self-embedding of the ring V. The $K[\Phi]$-module K is also a module over the subring $V[\Phi]$ of $K[\Phi]$. Let $U \subseteq V$. Then U is a $V[\Phi]$-submodule of V if and only if U is an ideal of V that is closed under ϕ. If $\gamma \in \Gamma$ satisfies $\gamma \geq 0$ and $\phi(\gamma) \geq \gamma$, then V_γ is a $V[\Phi]$-submodule of K. In particular, V and V_γ for $\gamma \geq g(0)$ where g is a modulus of growth for ϕ, are $V[\Phi]$-submodules of K. So every small enough ball in K is a $V[\Phi]$-submodule of K.

To transfer the structure given by the predicates in $\mathcal{P}$ to V, we have to allow translates of those predicates. Consider $\tilde{\mathcal{P}} = K^\times \times \mathcal{P}$ as a set of unary predicate symbols. The symbol (μ, P) is denoted by μP. We regard V as a $\sigma_{V[\Phi],\tilde{\mathcal{P}}}$-structure by interpreting μP as $(\mu P(K)) \cap V$. If U is a $V[\Phi]$-submodule of V, then let

$$\tilde{\mathcal{P}}_U := \left\{ Q \in \tilde{\mathcal{P}} \;\middle|\; Q(V) = \{ 0 \} \text{ or } U \subseteq Q(V) \right\} .$$

Note that for $V[\Phi]$-submodules $U_1 \subseteq U_0$ of V we have $\tilde{\mathcal{P}}_{U_1} \supseteq \tilde{\mathcal{P}}_{U_0}$ and that every finite subset of $\tilde{\mathcal{P}}$ is contained in $\tilde{\mathcal{P}}_U$ for some $V[\Phi]$-submodule U of V. Let U be a $V[\Phi]$-submodule of V. We regard V/U as a $\sigma_{V[\Phi],\tilde{\mathcal{P}}_U}$-structure by interpreting $Q \in \tilde{\mathcal{P}}_U$ as $Q(V)/U$, if $U \subseteq Q(V)$, and as U/U, if $Q(V) = \{0\}$. Alternatively, we can regard V/U as $(V/U)[\Phi] \simeq V[\Phi]/U[\Phi]$-module (see Remark 2.4) with the same predicates as above. The distinction is not important, since $f \in V[\Phi]$ acts on V/U the same as $f/U[\Phi]$.

Since $\phi(V_{>0}) \subseteq V_{>0}$, the self-embedding ϕ induces a self-embedding of the residue field k, which is also denoted by ϕ. So k becomes a k$[\Phi]$-module with respect to this self-embedding.

We will show in the next section, how to reduce certain questions about the $K[\Phi]$-module structure of K to questions about the $V[\Phi]$-module structure of V/U where U is a $V[\Phi]$-submodule of V. In Section 8.2, this will be specialized to the situation where $\Gamma = \mathbb{Z}$ and the residue field is embedded in K and mapped into itself by ϕ. In this situation, one can reduce the questions to the k$[\Phi]$-module structure of k.

8.1 The general case

We start with two simple lemmas about (additive) abelian groups. The first one will be used to compute indices of pp-definable sets and the second one in connection with model-completeness.

8.1 Lemma. *Let G be an abelian group and E, F, B_0, B_1 be subgroups of G such that*

1. $E \supseteq F$,
2. $B_0 \supseteq B_1$,
3. $E + B_0 = F + B_0$,
4. $E \cap B_1 = F \cap B_1$.

Let $E' = (E \cap B_0) + B_1$ and $F' = (F \cap B_0) + B_1$. Then $B_1 \subseteq F' \subseteq A' \subseteq B_0$ and $|E/F| = |E'/F'|$.

Proof. The relation $f = \{ (C, C') \in (E/F) \times (E'/F') \mid C \cap C' \neq \emptyset \}$ is an isomorphism of E/F onto E'/F'. $\square$

In the next lemma, the subscripts c, m and s stand for "complementary", "middle" and "small", respectively.

8.2 Lemma. *Let G be an abelian group and $A, A_c, A_m, A_s, B_0, B_1$ be subgroups of G such that*

1. $A + A_c = G$,
2. $A \cap A_c \subseteq B_0$,
3. $A \cap B_1 = A_s \cap B_1$,
4. $A \cap B_0 \subseteq A_m \subseteq A + B_1$.

Then for all $x \in G$, we have

$$(*) \quad x \in A \iff \forall x_c \in G\,((x - x_c \in A \wedge x_c \in A_c) \to$$
$$(x_c \in A_m \wedge \forall x_s \in G\,((x_c - x_s \in A \wedge x_s \in B_1) \to x_s \in A_s)))\,.$$

Proof. $\Longrightarrow$: Let $x_c \in A_c$ such that $x - x_c \in A$. Then $x_c \in A$, so $x_c \in A \cap A_c \subseteq B_0$, and therefore $x_c \in A \cap B_0 \subseteq A_m$. Let $x_s \in B_1$ such that $x_c - x_s \in A$. Then $x_s \in A$, because $x_c \in A$, and so $x_s \in A \cap B_1 \subseteq A_s$.

$\Longleftarrow$: Let $x \in G$ such that the right hand side of (*) holds. Since $G = A + A_c$, there are $x_1 \in A$ and $x_c \in A_c$ such that $x = x_1 + x_c$. So $x - x_c = x_1 \in A$. It follows that $x_c \in A_m$ and

$$(+) \qquad \forall x_s \in G\,((x_c - x_s \in A \wedge x_s \in B_1) \to x_s \in A_s)\,.$$

Since $A_m \subseteq A + B_1$, there are $x_2 \in A$ and $x_s \in B_1$ with $x_c = x_2 + x_s$, so $x_c - x_s \in A$. By (+), we can conclude $x_s \in A_s$, so $x_s \in A_s \cap B_1 \subseteq A$. Thus we get $x_c \in A$ and then $x \in A$. $\square$

8.3 Corollary. *In the situation of the previous lemma, suppose that $(G, +, -, 0)$ is the reduct of some structure $\hat{G}$ such that A, A_c, B_1 are definable in $\hat{G}$ by existential formulas and A_m, A_s by universal formulas. Then A is definable in $\hat{G}$ by a universal formula.*

Next, we give a slightly technical statement about preservation of universal definability under interpretations. For the definition of an interpretation and basic facts, see [Ho], sections 5.3 and 5.4 .

8.4 Lemma. *Suppose L_i is a language and M_i an L_i-structure for $i = 0, 1$. Let Δ be a d-dimensional interpretation of M_0 in M_1 consisting of*

1. *an L_1-formula $\partial_\Delta(y_1, \ldots, y_d)$,*
2. *a map that assigns to each unnested atomic formula $\tau(x_1, \ldots, x_n)$ of L_0 an L_1-formula $\tau_\Delta(y_{11}, .., y_{1d}, \ldots, y_{n1}, .., y_{nd})$, and*
3. *a surjective map $f_\Delta : \partial_\Delta(M_1) \longrightarrow M_0$,*

such that for each $\tau(x_1, \ldots, x_n)$ as in part 2 and all $(a_1, \ldots, a_n) \in \partial_\Delta(M_1)^n$, one has

$$M_0 \models \tau(f_\Delta(a_1), \ldots, f_\Delta(a_n)) \iff M_1 \models \tau_\Delta(a_{11}, .., a_{1d}, \ldots, a_{n1}, .., a_{nd})\,,$$

where $a_i = (a_{i1}, \ldots, a_{id})$ for $i = 1, \ldots, n$.

We assume the following:

(i) *For every $\tau(x_1, \ldots, x_n)$ as in part 2, the formula $\tau_\Delta(y_{11}, .., y_{1d}, \ldots, y_{n1}, .., y_{nd})$ is equivalent in M_1 to an existential and to a universal formula.*

(ii) $\partial_\Delta(y_1,\dots,y_d)$ *is equivalent in* M_1 *to an existential formula.*

(iii) $\partial_\Delta(y_1,\dots,y_d)$ *is equivalent in* M_1 *to a universal formula.*

Then for every subset A_0 *of* M_0^n *definable by a universal formula in* M_0*, the set*

$$A_1 := \{\, (a_{11},..,a_{1d},\dots,a_{n1},..,a_{nd}) \in \partial_\Delta(M_1)^n \mid (f_\Delta((a_{11},..,a_{1d})),\dots,f_\Delta((a_{n1},..,a_{nd}))) \in A_0 \,\}$$

is definable by a universal formula in M_1*.*

Proof. Let $\psi(x_1,\dots,x_n)$ be a universal formula of L_0 defining $A_0 \subseteq M_0^n$ in M_0. By the proof of [Ho], theorem 5.3.2, using assumptions (i) and (ii) above, we can find a universal L_1-formula $\psi'(y_{11},..,y_{1d},\dots,y_{n1},..,y_{nd})$ such that for all $(a_1,\dots,a_n) \in \partial_\Delta(M_1)^n$, with $a_i = (a_{i1},..,a_{id})$ for $i = 1,\dots,n$, we have

$$M_0 \models \psi(f_\Delta(a_1),\dots,f_\Delta(a_n)) \iff M_1 \models \psi'(a_{11},..,a_{1d},\dots,a_{n1},..,a_{nd})\,.$$

Thus the set A_1 is defined in M_1 by the L_1-formula

$$\psi'(y_{11},..,y_{1d},\dots,y_{n1},..,y_{nd}) \wedge \bigwedge_{i=1}^{n} \partial_\Delta(y_{i1},..,y_{id})\,,$$

which by assumption (iii) is equivalent to a universal formula in M_1. □

8.5 Remark. Let U be a $V[\Phi]$-submodule of the valuation ring V. Suppose $V \subseteq K$ is definable by a formula $\tau_V(x)$ in K. Let $\mu \in K$ be such that $\mu U = V$. We exhibit a 1-dimensional interpretation Δ of the $\sigma_{V[\Phi],\tilde{\mathcal{P}}_U}$-structure V/U in K: Define $f_\Delta : V \longrightarrow V/U\,,\ w \mapsto w/U$ and for all distinct variables x, y, z, set

$$\begin{aligned}
\partial_\Delta(x) &:= \tau_V(x)\,,\\
(x = y)_\Delta &:= \tau_V(\mu\cdot(x-y))\,,\\
(x + y = z)_\Delta &:= \tau_V(\mu\cdot(x+y-z))\,,\\
(g\cdot x = y)_\Delta &:= \tau_V(\mu\cdot(g\cdot x - y)) \quad \text{for } g \in V[\Phi],\\
Q(x)_\Delta &:= Q(x) \quad \text{for } Q \in \tilde{\mathcal{P}}_U \text{ with } U \subseteq Q(V),\\
Q(x)_\Delta &:= \tau_V(\mu\cdot x) \quad \text{for } Q \in \tilde{\mathcal{P}}_U \text{ with } Q(V) = \{\,0\,\}.
\end{aligned}$$

If V is definable in K by an existential and by a universal formula, then the interpretation Δ satisfies the hypothesis of Lemma 8.4.

8.6 Definition. Let $\lambda \in K^\times$. Let I be a finite (index) set and $P = (P_i)_{i\in I} \in \mathcal{P}^I$.

Let Z be a finite set of variables and $M \in K[\Phi]^{I\times Z}$. If M is not the zero matrix and $\mu \in K^\times$ is such that $-v(\mu)$ is equal to the minimum of the valuations of all coefficients of the polynomials

$M(i,z)\lambda^{-1} \in K[\Phi]$ where $i \in I$, $z \in Z$, then setting $\tilde{M} := \mu M \lambda^{-1}$ we have $\tilde{M} \in V[\Phi]^{I\times Z}$ and we call $\tilde{M}$ a **λ-translate of M with respect to μ**. If M is the zero matrix, the zero matrix in $V[\Phi]^{I\times Z}$ is called a **λ-translate of M with respect** 1.

For any λ-translate $\tilde{M}$ of M with respect to $\mu \in K^\times$, the $\sigma_{V[\Phi],\tilde{\mathcal{P}}}$-formula $\tilde{M} \cdot Z \in \tilde{P}$ with $\tilde{P} \in \tilde{\mathcal{P}}^I$ defined by $\tilde{P}_i = \mu P_i$ for $i \in I$ is called a **λ-translate of the $\sigma_{K[\Phi],\mathcal{P}}$-formula** $M \cdot Z \in P$.

Let X and Y be finite disjoint sets of variables, $Z := X \cup Y$ and $M \in K[\Phi]^{I\times Z}$. Let $\tau(Z)$ be the formula $M \cdot Z \in P$. For any λ-translate $\tilde{\tau}(Z)$ of $\tau(Z)$, the $\sigma_{V[\Phi],\tilde{\mathcal{P}}}$-formula $\exists Y\, \tilde{\tau}(X,Y)$ is called a **λ-translate of** $\exists Y\, \tau(X,Y)$.

8.7 Remark. Let $\lambda \in K^\times$. Let I be finite (index) set and $P \in \mathcal{P}^I$. Let Z be a finite set of variables and $M \in K[\Phi]^{I\times Z}$. Let $\tau(Z)$ be the formula $M \cdot Z \in P$. The formula $\tau(Z)$ has a λ-translate. If $Z = X \dot\cup Y$, then the formula $\exists Y (M \cdot Z \in P)$ has a λ-translate. So every special pp-formula has a λ-translate.

If $\tilde{\tau}(Z)$ is any λ-translate of $\tau(Z)$, then for all $w \in K^Z$ such that $\lambda w \in V^Z$, we have

$$K \models \tau(w) \iff V \models \tilde{\tau}(\lambda w)\,.$$

For special pp-formulas in general, we cannot expect to have such a nice relation of the solution sets of τ and a λ-translate $\tilde{\tau}$ of it, but a weakened version holds:

8.8 Lemma. *Let I be a finite (index) set and $P = (P_i)_{i\in I} \in \mathcal{P}^I$. Let X, Y be disjoint finite sets of variables and $M \in K[\Phi]^{I\times(X\cup Y)}$.*

1. *Let B be a ball in K^X. Then there exists $\gamma \in \Gamma$ such that for all $\lambda \in K^\times$ with $v(\lambda) \geq \gamma$ and all λ-translates $\tilde{\tau}(X)$ of $\tau(X) = \exists Y\, (M \cdot (X \cup Y) \in P)$, we have*

$$\big(w \in B \text{ and } K \models \tau(w)\big) \implies \big(\lambda w \in V^X \text{ and } V \models \tilde{\tau}(\lambda w)\big)\,.$$

2. *Let U be a ball in K^X and $\lambda \in K^\times$. Then there exists a ball $U' \subseteq V$ such that U' is a $V[\Phi]$-submodule of V, every λ-translate $\tilde{\tau}(X)$ of $\tau(X) = \exists Y\, (M \cdot (X \cup Y) \in P)$ is a $\tilde{\mathcal{P}}_{U'}$-formula and*

$$w \in K^X \text{ and } \lambda w \in V^X \text{ and } V/U' \models \tilde{\tau}(\lambda w) \implies \text{there exists } w' \in K^X \text{ such that } K \models \tau(w') \text{ and } w - w' \in U\,.$$

Proof. For the first part, we use Lemma 7.7 to find a ball B' in K^Y such that for all $w \in B$ the following holds: If there exists $u \in K^Y$ such that $K \models M \cdot (w^\frown u) \in P$, then there exists $u' \in B'$ such that $K \models M \cdot (w^\frown u') \in P$.

Pick $\lambda_0 \in K^\times$ with $\lambda_0 B \subseteq V^X$ and $\lambda_0 B' \subseteq V^Y$, and set $\gamma = v(\lambda_0)$. The statement now follows from the previous remark, since for all $\lambda \in K$ with $v(\lambda) \geq \gamma$ we have $\lambda B \subseteq V^X$ and $\lambda B' \subseteq V^Y$.

For the second part, let U be a ball in K^X and $\lambda \in K^\times$. Let $\mu \in K^\times$ be such that $\tilde{M}$ is a λ-translate of M with respect to μ. Note that all such μ have the same valuation. Let $\tau(X) = \exists Y\, (M \cdot (X \cup Y) \in P)$ and $\tilde{\tau}(X) = \exists Y\, (\tilde{M} \cdot (X \cup Y) \in \tilde{P})$ with $\tilde{P} \in \tilde{\mathcal{P}}^I$ defined by $\tilde{P}_i = \mu P_i$ for $i \in I$.

Pick a ball U_0 such that for all $i \in I$, we have $P_i \in \tilde{\mathcal{P}}_{U_0}$. Let $I_0 := \left\{\, i \in I \mid \tilde{P}_i(V) = \{\,0\,\} \right\}$ and $I_1 := I \setminus I_0$. Pick a ball U_1 in $K^{X \cup Y}$ such that $U_1 \subseteq U \times K^Y$ and $M_1 \cdot U_1 \subseteq U_0^{I_1}$ where $M_1 = M\restriction_{I_1 \times (X \cup Y)}$. Applying Proposition 7.35 to the matrix $M_0 = M\restriction_{I_0 \times (X \cup Y)}$, we obtain a ball U_2 in K^{I_0} such that for all $a \in K^{X \cup Y}$ one has

$$M_0 \cdot a \in U_2 \implies \exists a' \in K^{X \cup Y}\, (a - a' \in U_1 \wedge M_0 \cdot a' = 0)\,.$$

Now choose a ball U' in K that is a $V[\Phi]$-submodule of V such that $U'' := \mu^{-1} U'$ satisfies $U'' \subseteq U_0$ and $U''^{I_0} \subseteq U_2$, and note that the choice of U' can be made independent of the choice of μ. Also note that $\tilde{\tau}(X)$ is a $\tilde{\mathcal{P}}_{U'}$-formula.

Let $w \in K^X$ such that $\lambda w \in V^X$ and $V/U' \models \tilde{\tau}(\lambda w)$. Then there exist $w_Y \in K^Y$ such that $\lambda w_Y \in V^Y$ and $\tilde{M} \cdot (\lambda a) \in \prod_{i \in I}((\mu P_i \cap V) + U')$ for $a = w^\frown w_Y$, so $M \cdot a \in \prod_{i \in I}(P_i + U'')$.

For $i \in I_0$, we have $P_i + U'' = U''$, so $M_0 \cdot a \in U''^{I_0} \subseteq U_2$. Therefore we can pick $a' \in K^{X \cup Y}$ such that $a - a' \in U_1$ and $M_0 \cdot a' = 0$. Since $a - a' \in U_1$, we obtain $M_1 \cdot (a - a') \in U_0^{I_1} \subseteq \prod_{i \in I_1} P_i$. For $i \in I_1$, we have $P_i + U'' = P_i$, so $M_1 \cdot a \in \prod_{i \in I_1} P_i$, and thus $M_1 \cdot a' \in \prod_{i \in I_1} P_i$.

We have shown that $M \cdot a' \in \prod_{i \in I} P_i$, so choosing w' as the projection of a' on the components in X, we have $K \models \tau(w')$ and $w - w' \in U$, because $U_1 \subseteq U \times K^Y$. □

8.9 Theorem. *Suppose that for all sufficiently small balls U, the $\sigma_{V[\Phi],\tilde{\mathcal{P}}_U}$-structure V/U is model-complete. Also assume that the subset V of K is definable by a pp-formula and by a universal formula in the $\sigma_{K[\Phi],\mathcal{P}}$-structure K. Then the $\sigma_{K[\Phi],\mathcal{P}}$-structure K is model-complete.*

Proof. By Theorem 5.13, it suffices to show that every pp-definable set A in the structure K is defined by a universal formula. So let X be a finite set of variables and $A \subseteq K^X$ be pp-definable. By Proposition 7.19, there exists a pp-definable set $A_c \subseteq K^X$ such that $A + A_c = K^X$ and $A \cap A_c$ is bounded. Let B_0 be a ball in K^X such that $A \cap A_c \subseteq B_0$. Let B be a basis of K over $\phi(K)$. By Proposition 7.27, there exists $c \in \mathbb{N}$ and a separable left submodule N of $K[\Phi]^J$ with $J := X \times B^c$ such that $T^c(A) \overset{0}{=} \mathrm{Ann}(N)$ where T^c is the canonical bijection $K^X \longrightarrow K^{X \times B^c} = K^J$ determined by the basis B. The set $\mathrm{Ann}(N)$ and the graph of the function T^c can be defined by conjunctions of atomic formulas. Thus the set $A_s := (T^c)^{-1}(\mathrm{Ann}(N))$ can be defined by a universal formula. Because $T^c(A) \overset{0}{=} \mathrm{Ann}(N)$, we have $A \overset{0}{=} A_s$, so there exists a ball B_1 in K^X such that $A \cap B_1 = A_s \cap B_1$. Because V is defined by an existential formula, so is B_1. Now let $\tau(X)$ be a special pp-formula that defines A. By part 1 of Lemma 8.8, there exists $\lambda \in K^\times$ and a λ-translate $\tilde{\tau}(X)$ of $\tau(X)$ such that

$$(\subseteq) \qquad \big(w \in B_0 \text{ and } K \models \tau(w)\big) \implies \big(\lambda w \in V^X \text{ and } V \models \tilde{\tau}(\lambda w)\big)$$

for all w. By part 2 of Lemma 8.8, there exists a ball $U' \subseteq V$ that is a $V[\Phi]$-submodule such that $\tilde{\tau}(X)$ is a $\tilde{\mathcal{P}}_{U'}$-formula and

$$(\supseteq) \quad w \in K^X \text{ and } \lambda w \in V^X \text{ and } V/U' \models \tilde{\tau}(\lambda w) \\ \implies \text{there exists } w' \in K^X \text{ such that } K \models \tau(w') \text{ and } w - w' \in B_1$$

for all w. By shrinking U' if necessary (noting that $(\supseteq)$ is still true), we can assume that the $\tilde{\mathcal{P}}_{U'}$-structure V/U' is model-complete.

Let

$$A^* := \{ \, w \in V^X \mid V/U' \models \tilde{\tau}(w/U') \, \} \, ,$$
$$A_m := \{ \, w \in K^X \mid \lambda w \in A^* \, \} \, .$$

Then $(\subseteq)$ yields $A \cap B_0 \subseteq A_m$ and $(\supseteq)$ yields $A + B_1 \supseteq A_m$. Since V/U' is model-complete, there is a universal $\tilde{\mathcal{P}}_{U'}$-formula $\psi(X)$ that defines the set $\tilde{\tau}(V/U')$. By Remark 8.5, the set A^* is definable by a universal formula in K, and thus the same holds for A_m. Now apply Corollary 8.3. □

8.10 Corollary. *If the residue field* k *is finite,* $\Gamma = \mathbb{Z}$ *and* $\mathcal{P}$ *contains a predicate whose interpretation is* V*, then* K *is model-complete.*

8.11 Result. *Let* $\alpha(X), \beta_0(X)$ *be pp-formulas in the signature* $\sigma_{K[\Phi],\mathcal{P}}$*. Let* $\beta(X) := \alpha(X) \wedge \beta_0(X)$*. Here we outline a procedure to reduce the computation of the index* $|\alpha(K)/\beta(K)|$ *to the computation of an index* $|\tilde{\alpha}(V/U)/(\tilde{\alpha}(V/U) \cap \tilde{\beta}(V/U))|$ *in the structure* V/U *for some submodule* U *of* V *and some pp-formulas* $\tilde{\alpha}(X), \tilde{\beta}(X)$ *in the signature* $\sigma_{V[\Phi],\tilde{\mathcal{P}}_U}$*.*

First, compute $\dim_\infty \alpha(X)$ *and* $\dim_\infty \beta(X)$*. If* $\dim_\infty \beta(X) < \dim_\infty \alpha(X)$*, then* $|\alpha(K)/\beta(K)| = \infty$*, because* $\alpha(K) \subseteq \beta(K) + B_0$ *for no ball* B_0 *in* K^X*, and* Γ *is non-trivial. Suppose* $\dim_\infty \beta(X) = \dim_\infty \alpha(X)$*. Then we can determine a ball* B_0 *in* K^X *such that*

$$\alpha(K) + B_0 = \beta(K) + B_0 \, .$$

Next, compute $\dim_0 \alpha(X)$ *and* $\dim_0 \beta(X)$*. If* $\dim_0 \beta(X) < \dim_0 \alpha(X)$*, then* $|\alpha(K)/\beta(K)| = \infty$*, because* $\alpha(K) \cap B_1 \subseteq \beta(K)$ *for no ball* B_1 *in* K^X*, and* Γ *is non-trivial. Suppose* $\dim_0 \beta(X) = \dim_0 \alpha(X)$*. Then we can determine a ball* B_1 *in* K^X *such that*

$$\alpha(K) \cap B_1 = \beta(K) \cap B_1 \, .$$

Without loss of generality, we may assume that $\mathcal{P}$ *contains symbols* V_γ *for* $\gamma \in \Gamma$ *that are interpreted as the ball* V_γ *in* K*, since the addition of such symbols does not introduce additional structure in the* $\sigma_{V[\Phi],\tilde{\mathcal{P}}}$*-structure* V*. So* B_0 *and* B_1 *are definable by a conjunction of atomic formulas in* K *and* $(\alpha(K) \cap B_0) + B_1$ *and* $(\beta(K) \cap B_0) + B_1$ *are definable by pp-formulas, say by special pp-formulas* $\alpha'(X)$ *and* $\beta'(X)$*. By Lemma 8.1, we have* $|\alpha(K)/\beta(K)| = |\alpha'(K)/\beta'(K)|$*. So* *we can replace $\alpha(X)$ by $\alpha'(X)$ and $\beta(X)$ by $\beta'(X)$.*

By Lemma 8.8, part 1, there exists $\lambda \in K^\times$ and λ-translates $\tilde{\alpha}(X)$ of $\alpha(X)$ and $\tilde{\beta}(X)$ of $\beta(X)$ such that for $\tau = \alpha, \beta$ and the corresponding λ-translate $\tilde{\tau} = \tilde{\alpha}, \tilde{\beta}$ we have

$$(\subseteq) \qquad \big(w \in B_0 \text{ and } K \models \tau(w)\big) \implies \big(\lambda w \in V^X \text{ and } V \models \tilde{\tau}(\lambda w)\big).$$

By Lemma 8.8, part 2, there exists a ball $U \subseteq V$ that is a $V[\Phi]$-submodule such that $U^X \subseteq \lambda B_1$ and for $\tau = \alpha, \beta$ the corresponding λ-translate $\tilde{\tau} = \tilde{\alpha}, \tilde{\beta}$ is a $\tilde{\mathcal{P}}_U$-formula and

$$(\supseteq) \quad w \in K^X \text{ and } \lambda w \in V^X \text{ and } V/U \models \tilde{\tau}(\lambda w)$$
$$\implies \text{there exists } w' \in K^X \text{ such that } K \models \tau(w') \text{ and } w - w' \in B_1 .$$

Let $h : \lambda^{-1} V^X \longrightarrow (V/U)^X$ be the group homomorphism induced by multiplication by λ. Let $\tau = \alpha, \beta$ and $\tilde{\tau} = \tilde{\alpha}, \tilde{\beta}$ be the corresponding λ-translate. Then the statement ($\subseteq$) yields $\tau(K) \cap B_0 \subseteq h^{-1}(\tilde{\tau}(V/U))$. Since $\tau(K) \subseteq B_0$, we have $\tau(K) \subseteq h^{-1}(\tilde{\tau}(V/U)$. The statement ($\supseteq$) yields $\tau(K) + B_1 \supseteq h^{-1}(\tilde{\tau}(V/U))$. Since $\tau(K) \supseteq B_1$, we have $\tau(K) \supseteq h^{-1}(\tilde{\tau}(V/U))$. We obtain $\tau(K) = h^{-1}(\tilde{\tau}(V/U))$.

Since the image of h is $(V/U)^X$ and the kernel of h is contained in $B_1 \subseteq \tau(K)$, we can conclude that $|\alpha(K)/\beta(K)| = |\tilde{\alpha}(V/U)/\tilde{\beta}(V/U)|$.

8.2 The case $\Gamma = \mathbb{Z}$, residue field embedded

In this section, we assume that $\Gamma = \mathbb{Z}$. By Remark 7.3, part 6, one does not have to assume in this case that for every finite set X and finitely generated submodule M of $K[\Phi]^X$, the set $M \cdot K$ has the weak optimal approximation property modulo large balls, since it is trivially true.

We also assume that V has a subfield k' such that $\phi(\mathrm{k}') \subseteq \mathrm{k}'$ and the factor map $V \longrightarrow V/V_{>0} = \mathrm{k}$ restricted to k' is an isomorphism of fields.

We identify k' with k via this isomorphism and simply write k for both fields.

In this situation, one can explicitly say what form the $(V/U)[\Phi]$-modules V/U have. This is established in the lemma after the following definition.

8.12 Remark/Definition. Let L be a field and ϕ be a self-embedding of L. As before we consider L as a left $L[\Phi]$-module via $\Phi \cdot \mu = \phi(\mu)$ for $\mu \in L$.

Let $L[x]$ be the ordinary polynomial ring in one variable over L. Let $f \in L[x]$. We can extend ϕ to a ring homomorphism ϕ_f of the ring $L[x]$ by setting $\phi_f(x) = f$. Every extension of ϕ to a ring homomorphism of $L[x]$ has this form. We obtain a twisted polynomial ring $L[x][\Phi_f]$ over $L[x]$, and $L[x]$ becomes a left $L[x][\Phi_f]$-module via $(g\Phi_f^0) \cdot h := gh$ and $\Phi_f \cdot g := \phi_f(g)$ for $g, h \in L[x]$.

Suppose $f \in xL[x]$. For every n, the ideal $I = x^n L[x]$ of satisfies $\phi_f(I) \subseteq I$, so is a $L[x][\Phi_f]$-submodule of $L[x]$. Also $I[\Phi_f]$ is an ideal of $L[x][\Phi_f]$ and $L[x][\Phi_f]/I[\Phi_f]$ is canonically isomorphic to

$(L[x]/x^n)[\Phi_f]$. On the other hand, every ring endomorphism of $L[x]/x^n \supseteq L$ extending $\phi : L \longrightarrow L$ is induced by ϕ_f for some $f \in xL[x]$ with $\deg f < n$.

8.13 Lemma. *Let $m := \phi(1)$ where $1 \in \Gamma = \mathbb{Z}$. Note that $m \geq 2$ follows from the assumption that ϕ has a modulus of growth. Suppose $U \subseteq V$ is a ball. Then $U = V_n$ for some n, and U is a $V[\Phi]$-submodule of V. The ring V/U is isomorphic to $\mathrm{k}[x]/x^n$ over k, and for any such isomorphism, the map induced by ϕ on V/U corresponds to the map induced by ϕ_f on $\mathrm{k}[x]/x^n$ for some $f \in x^m\mathrm{k}[x]$ with $\deg f < n$. In addition, identifying these two rings via such an isomorphism, the modules V/U and $\mathrm{k}[x]/x^n$ are isomorphic.*

Proof. Recall, that we view k as subring of V. Let $t \in K$ with $v(t) = 1$. The subring $\mathrm{k}[t]$ of V is isomorphic to the polynomial ring $\mathrm{k}[x]$ (mapping t to x). It is easy to see that $V/U = V/t^n$ is isomorphic to $\mathrm{k}[t]/t^n$. By the previous remark, the ring homomorphism corresponding to the ring homomorphism induced by ϕ on $\mathrm{k}[t]/t^n$ is induced by ϕ_f for some $f \in x\mathrm{k}[x]$ with $\deg f < n$. Since $\phi(t) \in t^mV$, we have $f \in x^m\mathrm{k}[x]$. □

8.14 Assumption. For the rest of the section, $\mathcal{P}$ contains just the two symbols V_0 and V_∞, where in K the symbol V_0 is interpreted as V and V_∞ as $\{0\}$. Let $m := \phi(1)$.

8.15 Corollary.

1. *Suppose that for all $n \in N$ and $f \in x^2\mathrm{k}[x]$, the $(\mathrm{k}[x]/x^n)[\Phi_f]$-module $\mathrm{k}[x]/x^n$ is model-complete in the signature $\sigma_{(\mathrm{k}[x]/x^n)[\Phi_f]}$, then the $\sigma_{K[\Phi],\mathcal{P}}$-structure K is model-complete.*

2. *The computation of pp-indices in the structure K can be reduced to those in the $\sigma_{(\mathrm{k}[x]/x^n)[\Phi_f]}$-structures $\mathrm{k}[x]/x^n$ for $n \in \mathbb{N}$, $f \in x^2\mathrm{k}[x]$.*

Proof. Note that for any ball $U \subseteq V$ and $P \in \tilde{\mathcal{P}}_U$, the set $P(V/U)$ is definable by an atomic formula in the $\sigma_{V[\Phi]}$-structure V/U.

So part 1 follows from Theorem 8.9 and part 2 follows from Result 8.11. □

In the following, we try to state the previous corollary in terms of the $\mathrm{k}[\Phi]$-module k instead of the $(\mathrm{k}[x]/x^n)[\Phi_f]$-module $\mathrm{k}[x]/x^n$ where $f \in x^2\mathrm{k}[x]$ and $n \in \mathbb{N}$. This is possible, since the two structures are bi-interpretable for $n \geq 1$ in a nice way. As a preparation for the model-completeness part, we state the following lemma.

8.16 Lemma. *Suppose L_i is a language and M_i an L_i-structure for $i = 0, 1$. For $i = 0, 1$, let Δ_i be an d_i-dimensional interpretation of M_i in M_{1-i}, which consists of*

1. *an L_{1-i}-formula $\partial_{\Delta_i}(y_1, \ldots, y_{d_i})$,*

2. *a map that assigns to each unnested atomic formula $\tau(x_1, \ldots, x_n)$ of L_i an L_{1-i}-formula $\tau_{\Delta_i}(y_{11}, \ldots, y_{1d_i}, \ldots, y_{n1}, \ldots, y_{nd_i})$, and*

3. *a surjective map $f_{\Delta_i} : \partial_{\Delta_i}(M_{1-i}) \longrightarrow M_i$,*

such that for each $\tau(x_1,\ldots,x_n)$ *as in part 2 and all* $(a_1,\ldots,a_n)\in\partial_{\Delta_i}(M_{1-i})^n$*, one has*

$$M_i \models \tau(f_\Delta(a_1),\ldots,f_{\Delta_i}(a_n)) \iff M_{1-i}\models \tau_\Delta(a_{11},..,a_{1d_i},\ldots,a_{n1},..,a_{nd_i}),$$

where $a_j=(a_{j1},..,a_{jd_i})$ *for* $j=1,\ldots,n$.

Also suppose that:

(i) *The composite interpretation* $\Delta_1\circ\Delta_0$ *of* M_0 *in* M_0 *is homotopic to the identity interpretation on* M_0 *via some existential formula, i.e. there exists an existential* L_0*-formula*

$$\chi(x,y_{11},..,y_{1d_1},\ldots,y_{d_01},..,y_{d_0d_1})$$

that induces an isomorphism of M_0 *onto* $(\Delta_1\circ\Delta_0)(M_0)$.

(ii) $\partial_{\Delta_1}(y_1,\ldots,y_{d_1})$ *is equivalent in* M_0 *to an existential formula.*

(iii) *For every* $\tau(x_1,\ldots,x_n)$ *as in part 2 with* $i=1$*, the formula* $\tau_{\Delta_1}(y_{11},..,y_{1d_1},\ldots,y_{n1},..,y_{nd_1})$ *is equivalent in* M_0 *to an existential and to a universal formula.*

(iv) M_1 *is model-complete.*

Then M_0 *is model-complete.*

Proof. Suppose $\alpha(x_1,\ldots,x_n)$ is an L_0-formula. It suffices to show that $\alpha(x_1,\ldots,x_n)$ is equivalent to an existential formula in M_0.

By the homotopy assumption (i), we know that $\alpha(x_1,\ldots,x_n)$ is equivalent to

$$\exists y_{111},..,y_{11d_1},\ldots,y_{nd_01},..,y_{nd_0d_1}\\ (\bigwedge_{j=1}^n \chi(x_j,y_{j11},..,y_{j1d_1},\ldots,y_{jd_01},..,y_{jd_0d_1})\wedge(\alpha_{\Delta_0})_{\Delta_1}(y_{111},..,y_{11d_1},\ldots,y_{nd_01},..,y_{nd_0d_1})). \tag{8.1}$$

where $\chi(x,y_{11},..,y_{1d_1},\ldots,y_{d_01},..,y_{d_0d_1})$ is an existential formula. So it suffices it show that $(\alpha_{\Delta_0})_{\Delta_1}(y_{111},..,y_{11d_1},\ldots,y_{nd_01},..,y_{nd_0d_1})$ is equivalent to an existential formula in M_0.

The formula $\alpha_{\Delta_0}(y_{11},..,y_{1d_0},\ldots,y_{n1},..,y_{nd_0})$ is equivalent to an existential formula

$$\beta(y_{11},..,y_{1d_0},\ldots,y_{n1},..,y_{nd_0})$$

in M_1, since M_1 is model-complete by assumption. By introducing additional existential quantifiers, we may assume that the atomic subformulas of β are unnested. The assumptions (ii) and (iii) yield that $\beta_{\Delta_1}(y_{111},..,y_{11d_1},\ldots,y_{nd_01},..,y_{nd_0d_1})$ is equivalent to an existential formula in M_0. □

8.17 Lemma. *Let* $n\geq 1$ *and* $g\in x^2\mathrm{k}[x]$*. There exist an* n*-dimensional interpretation* $\Delta_{(\mathrm{k}[x]/x^n)}$ *of the* $(\mathrm{k}[x]/x^n)[\Phi_g]$*-module* $\mathrm{k}[x]/x^n$ *in the* $\mathrm{k}[\Phi]$*-module* k *and a* 1*-dimensional interpretation* Δ_{k} *of* *the* $\mathrm{k}[\Phi]$*-module* k *in the* $(\mathrm{k}[x]/x^n)[\Phi_g]$*-module* $\mathrm{k}[x]/x^n$ *such that the assumptions of the previous lemma are satisfied for* $\Delta_0 = \Delta_{\mathrm{k}}$*,* $\Delta_1 = \Delta_{(\mathrm{k}[x]/x^n)}$ *and for* $\Delta_1 = \Delta_{\mathrm{k}}$*,* $\Delta_0 = \Delta_{(\mathrm{k}[x]/x^n)}$*.*

Proof. We may assume that $\deg g < n$. For $i < n$, pick $g_{i,j} \in \mathrm{k}$ for $j < n$ such that $g^i + \mathrm{k}[x]x^n = \sum_{j<n} g_{i,j}x^j + \mathrm{k}[x]x^n$.

For $\Delta_{(\mathrm{k}[x]/x^n)}$, use $f_{\Delta_{(\mathrm{k}[x]/x^n)}}$ defined by

$$f((\lambda_i)_{i<n}) := \sum_{i<n} \lambda_i x^i + \mathrm{k}[x]x^n$$

for $(\lambda_i)_{i<n} \in \mathrm{k}^n$, and

$$\begin{aligned}
(z = \Phi \cdot y)_{\Delta_{(\mathrm{k}[x]/x^n)}} &:= \bigwedge_{j<n} z_j = \sum_{i<n} (g_{i,j}\Phi) \cdot y_i \,, \\
(z = x \cdot y)_{\Delta_{(\mathrm{k}[x]/x^n)}} &:= z_0 = 0 \wedge \bigwedge_{i<n-1} z_{i+1} = y_i \,, \\
(z = \lambda \cdot y)_{\Delta_{(\mathrm{k}[x]/x^n)}} &:= \bigwedge_{i<n} z_i = \lambda y_i
\end{aligned}$$

for all variables y, z and $\lambda \in \mathrm{k}$.

To construct Δ_{k}, use the following: For $t, m \in \mathbb{N}$ with $f \in x^m\mathrm{k}[x]$ and $m^t \geq n$ and a basis B_t for k over $\phi^t(\mathrm{k})$, we have $(\mathrm{k}x^0)/x^n = \sum_{b\in B_t}(b\Phi^t) \cdot (\mathrm{k}[x]/x^n)$. Thus the subset $(\mathrm{k}x^0)/x^n$ is existentially definable in $\mathrm{k}[x]/x^n$. □

8.18 Corollary.

1. *Suppose that the* $\mathrm{k}[\Phi]$*-module* k *is model-complete in the signature* $\sigma_{\mathrm{k}[\Phi]}$*. Then the* $\sigma_{K[\Phi],\mathcal{P}}$ *structure* K *is model-complete.*

2. *The computation of pp-indices in the* $\sigma_{K[\Phi],\mathcal{P}}$*-structure* K *can be reduced to the computation of pp-indices in the* $\sigma_{\mathrm{k}[\Phi]}$*-structure* k.

Appendix A

Additional Conjectures

Here are some other mathematical statements which I believe are true, and hope to prove in the near future.

A.1 Conjecture. *In Theorem 7.17 we also have*

A) $\dim_\infty T_B(S) = [K : \phi(K)]\dim_\infty S$ *where* T_B *is the canonical bijection* $K^X \longrightarrow K^{X\times B}$.

B) If $E \in K^{X'\times X}$, *then for* $S' := E \cdot S$ *we have* $\dim_\infty S' \leq \dim_\infty S$.

A.2 Conjecture. *In Theorem 7.33 we also have*

A) $\dim_0 T_B(S) = [K : \phi(K)]\dim_0 S$ *where* T_B *is the canonical bijection* $K^X \longrightarrow K^{X\times B}$.

B) If $E \in K^{X\times X'}$ *has full row rank, then for* $S' := \left\{ a \in K^{X'} \;\middle|\; E \cdot a \in S \right\}$ *we have* $\dim_0 S' \geq \dim_0 S$.

A.3 Conjecture. *If* S *is a pp-definable set, then* $\dim_\infty S \leq \dim_0 S$.

A.4 Conjecture. *Let* k *be a field of characteristic* p, $K = \mathrm{k}((t))$ *and* ϕ *the Frobenius. Then the module* K *is not model-complete in the signature* $\sigma_{K[\Phi]}$. *If one adds a predicate for* V, *then it is model-complete if* k *is finite or satisfies the Kaplansky condition. (To say that* k *satisfies the Kaplansky condition means that* $f \cdot \mathrm{k} = \mathrm{k}$ *for every non-zero* $f \in \mathrm{k}[\Phi]$, *cf. [Ka], p.312, hypothesis A (1).)*

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Vita

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