Real holomorphy rings in function fields and their units

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17-th Hilbert Problem

Take $f \in \mathbb{R}[X_1, ..., X_n]$ with non-negative values. Are there $f_i \in \mathbb{R}(X_1, ..., X_n)$ such that

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Hilbert's 17th Problem was positively solved by E. Artin in 1927 by the theory of formally real fields.

Schülting's Problem (1987)

Schülting's Problem

Let $f, g \in \mathbb{R}[X]$ be of the same degree and without real zeros. Assume that $\frac{f}{g}$ is positive definite. Are there $f_i, g_i \in \mathbb{R}[X]$ without real zeros with $\deg f_i = \deg g_i$ such that

$$\frac{f}{g} = \sum_{i=1}^{k} \left(\frac{f_i}{g_i} \right)^2$$

for some natural number k?

Artin's solution for function fields

F - a formally real function field over a real closed field R

X - any smooth projective model of F

$X(R)$ - the set of rational points of X

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Can the real holomorphy ring of F be characterized in a similar way?

$\mathbb{R}$-places

F – a formally real field
P – an ordering of F
$A(P)$ – the convex hull of $\mathbb{Q}$ in F with respect to P
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- The ordering induced by P on the residue field $A(P)/I(P)$ is archimedean, therefore there is an embedding

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- The Baer-Krull Theorem says that every $\mathbb{R}$-place can be obtained in this way.

The space of orderings

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- compact,
- Hausdorff,
- totally disconnected.

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is onto, so we consider the quotient topology on $M(F)$ which is (D.W. Dubois, 1970):

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Orderings and $\mathbb{R}$-places of $\mathbb{R}(X)$

$$P_{+\infty} = \{\tfrac{f}{g} \mid \mathrm{lc}(f) \cdot \mathrm{lc}(g) > 0\}$$

$$P_{-\infty} = \{\tfrac{f}{g} \mid (-1)^{\deg f - \deg g} \cdot \mathrm{lc}(f) \cdot \mathrm{lc}(g) > 0\}$$

$$P_{a^+} = \{(x-a)^k \tfrac{f_1}{g_1} \mid f_1(a) g_1(a) > 0\}$$

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If F is a function field over a totally archimedean field K with finite number of orderings and $\mathrm{trdeg}(F/K) = 1$, then $M(F)$ is a disjoint finite union of circles (R. Brown 1980).

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$(A, B) < (C, D)$ - cuts in R.
Corresponding orderings determine the same $\mathbb{R}$-place iff $B \cap C$ is a coset of a convex subgroup of R.
The space $M(R(X))$ is:

- connected,
- not metrizable,
- self-similar,
- of topological dimension 1.

$$\mathcal{X}(\mathbb{R}(X, Y)) \cong \dot{\bigcup}\{\mathcal{X}(R(Y))\,;\ R \text{ real closure of } \mathbb{R}(X)\}.$$

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The space $M(\mathbb{R}(X, Y))$ is:

- path-connected (R. Brown, J. Merzel),
- not metrizable (M. Marshall, M. Machura, K.K.),
- of topological dimension 2 and cohomological dimension 1 (T. Banakh).

The Key Theorem

Theorem

Let F be a function field over a nonarchimedean real closed field R with natural valuation v_0. Take $P \in \mathcal{X}(F)$ and:

$$a_1, \dots, a_m \in P,$$

$$a_{m+1}, \dots, a_n \in I(P).$$

Then there is a rational place $\lambda : F \to R \cup \{\infty\}$ such that

$$\lambda(a_i) > 0 \text{ for } i = 1, \dots, m,$$

$$\lambda(a_i) \in \mathcal{I}(\dot{R}^2) \text{ for } i = m+1, \dots, n.$$

$t := (t_1, ..., t_k)$ - a transcendence base of F over R.

$$F = R(t, y),$$

where y is algebraic over $R(t)$.

The Key Theorem - proof

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We obtain the following elementary sentences:

- $f(t, y) = 0$,
- $\frac{\delta f}{\delta Y}(t, y) \neq 0$,
- $g_i(t) \neq 0$ for $i = 1, ..., n$,
- $f_i(t, y) g_i(t) \in P$ for $i = 1, ..., m$,
- $v_P(f_i(t, y)) > v_P(g_i(t))$ for $i = m + 1, ..., n$.

Since R is nonarchimedean valued, we have that $(R, \dot{R}^2, v_0)$ is existentially closed in (F, P, v_P).

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- $f(t', y') = 0$,
- $\frac{\delta f}{\delta Y}(t', y') \neq 0$,
- $g_i(t') \neq 0$ for $i = 1, ..., n$,
- $f_i(t', y')g_i(t') > 0$ for $i = 1, ..., m$,
- $v_0(f_i(t', y')) > v_0(g_i(t'))$ for $i = m + 1, ..., n$.

Consider the field $R(X_1, ..., X_k)$ with some rational place $\lambda : R(X_1, ..., X_k) \longrightarrow R \cup \{\infty\}$.

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Let (K, λ) be the henselization of $(R(X_1, ..., X_k), \lambda)$. Choose $t^* = (t_1^*, ..., t_k^*) \in K^k$ such that $t_1^*, ..., t_k^*$ are algebraically independent and $v_\lambda(t_i' - t_i^*) > 0$.

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Now we embed F in K by sending $t \mapsto t^*$ and $y \mapsto y^*$ and we identify F with its image in K. Then we restrict λ to F.

Real places of F

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$$M_R(F) := \{\xi \in M(F) \mid \exists \lambda : F \to R \cup \{\infty\};\ \xi = \xi_R \circ \lambda\}$$

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Density

Basis for the topology of $M(F)$:

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There is $\lambda : F \to R \cup \{\infty\}$ such that

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Therefore

$$q_1 \leq \xi_R \circ \lambda(f_i) \leq q_2 \, ,$$

which shows that $\xi_R \circ \lambda$ is in $U(f_1, ..., f_n)$.

Proposition

$M_R(F)$ is dense in $M(F)$.

The relative real holomorphy ring $H(F|R)$ and topology on $M(F|R)$

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Basis for the Euclidean topology of $M(F|R)$:

$$U(f_1, \dots, f_n) := \{ \lambda \in M(F|R) \mid \lambda(f_i) > 0 \}, \quad f_i \in H(F|R)$$

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2) The topology induced on $M_R(F)$ via this bijection is equal to the subspace topology of $M_R(F) \subset M(F)$, i.e., the bijection is a topological embedding of $M(F|R)$ in $M(F)$.

Rational places and $\mathbb{R}$-places

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2) The topology induced on $M_R(F)$ via this bijection is equal to the subspace topology of $M_R(F) \subset M(F)$, i.e., the bijection is a topological embedding of $M(F|R)$ in $M(F)$.

3) All nonempty open sets in $M(F|R)$ and in $M(F)$ contain infinitely many places, i.e., the spaces $M(F|R)$ and $M(F)$ do not have isolated points.

Schülting's results

Theorem (H.W. Schülting)

Let F be a function field over $\mathbb{R}$ and let D be a set of real valuations of F. The following statements are equivalent:
1) For every regular projective model X of F and every point $x \in X$ there exists a valuation $v \in D$ with center x,
2) $H(F) = H(F|\mathbb{R}) = \bigcap_{v \in D} \mathcal{O}_v$.

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Proposition (H.W. Schülting)

There exists a minimal class D of real valuations such that

$$H(F) = \bigcap_{v \in D} \mathcal{O}_v$$

if and only if $tr.deg(F|\mathbb{R}) = 1$.

A characterization of $H(F)$

Proposition

Let F be a function field over a real closed field R. Then the real holomorphy ring $H(F)$ is the intersection of the valuation rings of $\mathbb{R}$-places belonging to $M_R(F)$:

$$H(F) = \bigcap_{\xi \in M_R(F)} \mathcal{O}_\xi \, .$$

A characterization of $H(F|B)$

$$H(F|B) := \bigcap\{\mathcal{O} \subseteq F \mid \mathcal{O} \text{ real valuation ring with } B \subseteq \mathcal{O}\}\,.$$

Note that $H(F) = H(F|H(R))$.

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Proposition

For every real valuation ring $B \subseteq R$, we have

$$H(F|B) = H(F).B = \bigcap_{\lambda \in M(F|R)} \mathcal{O}_{\pi_B \circ \lambda}.$$

A minimal representation

Theorem

Let B, C be real valuation rings of R such that $B \subsetneq C$. Then:

1) $H(F|B) \subsetneq H(F|C)$;

2) the following statements are equivalent for each subset $\mathcal{F}$ of $M(F|R)$:

(a) $H(F|B) = \bigcap_{\lambda \in \mathcal{F}} \mathcal{O}_{\pi_B \circ \lambda}$,

(b) $\mathcal{F}$ is dense in $M(F|R)$;

3) There is no representation of the form (a) with minimal $\mathcal{F}$.

A geometrical characterization of $H(F)$

X - any smooth projective model of F

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$$f \in H(F) \Rightarrow f(x) \in H(R) \text{ for every } x \in X_f .$$

Define

$$H_X := \{f \in F \mid f(x) \in H(R) \text{ for every } x \in X_f\} .$$

X_f is Zariski-open, so we have:

$$H(F) \subseteq H_X \subseteq H(F|R) .$$

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- Take $\lambda \in M(F|R)$ and the center $c(\lambda)$ in $X_0(R)$
- $f(c(\lambda)) = \lambda(f) \in H(R)$
- $\xi_R \circ \lambda(f) \neq \infty$

A geometrical characterization of $H(F)$

$$H(F) \subseteq \bigcap \{H_X \mid X \text{ smooth, real complete model of } F\}$$

- Take f in the intersection
- $f \in H(F|R)$, so $\lambda(f) \in R$ for every $\lambda \in M(F|R)$
- There is a smooth real complete model X_0 such that f is regular in every point of $X_0(R)$
- Take $\lambda \in M(F|R)$ and the center $c(\lambda)$ in $X_0(R)$
- $f(c(\lambda)) = \lambda(f) \in H(R)$
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The proposition above is not true for an archimedean real closed field R since in this case $H_X = F$ for every smooth real complete model X of F.

Totally positive units of the real holomorphy ring

$U(F)$ - the set of units of $H(F)$

$$U^+(F) := H(F) \cap \Sigma F^{*2}$$

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$$U^+(\mathbb{R}(X)) = \left\{ \frac{f}{g} \in \mathbb{R}(X) : \deg f = \deg g \text{ and } f, g \in \Sigma \dot{\mathbb{R}}[X]^2 \right\}$$

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Schülting's Problem

Let $f, g \in \mathbb{R}[X]$ be of the same degree and without real zeros. Assume that $\frac{f}{g}$ is positive definite. Are there $f_i, g_i \in \mathbb{R}[X]$ without real zeros with $\deg f_i = \deg g_i$ such that

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Generalization of Schülting's Problem

Is every totally positive unit of the real holomorphy ring a sum of squares of totally positive units?

The Pythagorean number $P(F)$ of a field F is the minimal natural number such that every element which is a sum of squares is a sum of (at most) $P(F)$ squares.

Theorem (Schmid 1994)

Let $a \in U^+(F)$. Then there exists a natural number $n \leq P(F)$ and elements $u_1, \ldots, u_{n+1} \in U^+(F)$ such that

$$a = \sum_{i=1}^{n+1} u_i^2.$$

Can we reduce $n+1$ to n?

If $P(F) = 1$ the hypothesis is true iff $M(F)$ is connected.

A gap

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$$a \prec b :\Leftrightarrow b - a \in \Sigma \dot{F}^2.$$

Take $a, b \in U^+(F)$ such that $a \prec b$. Is there $y \in F$ such that

$$a \prec y^4 \prec b?$$

A counterexample

Take:

$$a = (\frac{X^2}{X^2+1})^2 + 2^2 = \frac{5X^4+8X^2+4}{(X^2+1)^2} \in U^+(\mathbb{Q}(X))$$

$$b = \frac{5X^4+8X^2+4}{(X^2+1)^2} + (\frac{X^2}{X^2+1})^2 = \frac{6X^4+8X^2+4}{(X^2+1)^2} \in U^+(\mathbb{Q}(X)).$$

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Take the X-adic place λ_0.

Note that $\lambda_0(a) = \lambda_0(b) = 4$.

A counterexample

Assume that there exists $y \in U^+(\mathbb{Q}(X))$ such that

$$a \prec y^4 \prec b\,.$$

So, for every $P \in \mathcal{X}(F)$,

$$a <_P y^4 <_P b\,.$$

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Then

$$\lambda_0(a) \leq \lambda_0(y^4) \leq \lambda_0(b) \,,$$
$$4 \leq \lambda_0(y^4) \leq 4 \,.$$

Since $\lambda_0(y) \in \mathbb{Q}$ and

$$\lambda_0(y^4) = (\lambda_0(y))^4 = 4 \,,$$

we obtain a contradiction.

For $a \in H(F)$, the function

$$\hat{a} : M(F) \to \mathbb{R},$$

$$\hat{a}(\xi) = \xi(a)$$

is continuous.

Becker's approach

For $a \in H(F)$, the function

$$\hat{a} : M(F) \to \mathbb{R},$$

$$\hat{a}(\xi) = \xi(a)$$

is continuous. The map

$$H(F) \to C(M(F), \mathbb{R}),$$

$$a \mapsto \hat{a}$$

is a $\mathbb{Q}$-algebra homomorphism with dense image (by the Stone-Weierstraß Theorem).

Becker's approach

$$S^n(H(F)) := \{(a_0, ..., a_n) \in F^{n+1} \mid \sum a_i^2 = 1\}$$

For $a = (a_0, ..., a_n) \in S^n(H(F))$ consider the function

$$\widehat{a} : M(F) \longrightarrow S^n,$$

$$\widehat{a}(\xi) = (\xi(a_0), ...\xi(a_n)).$$

The function $\widehat{a}$ is continuous.

Becker's approach

$$\Phi : S^n(H(F)) \longrightarrow C(M(F), S^n)$$

$$\Phi(a) = \widehat{a}$$

Theorem (Becker)

The following conditions are equivalent:

1. $\overline{S^{n-1}(H(F))} = C(M(F), S^{n-1})$,
2. *if* $a = a_1^2 + \ldots + a_n^2 \in U^+(F) \Rightarrow a = b_1^2 + \ldots + b_n^2$ *for* $b_i \in U^+(F)$.

Becker's solution of Schülting's problem for $\mathbb{R}(X)$

- $M(\mathbb{R}(X)) \cong S^1$;
- $\overline{S^{n-1}(H)}$ contains all homotopy classes of $\hat{a}$, $a \in S^n(H)$;
- all homotopy classes of $C(S^1, S^1)$ are characterized by the degree of the functions;
- every continuous function $M(\mathbb{R}(X)) \longrightarrow S^1$ is homotopic to $\hat{f^d}$, where $f = (\frac{2X}{X^2+1}, \frac{X^2-1}{X^2+1})$.

Open problem - making Schmid's proof complete

Let R be a real closed field. Take $f, g \in U^+(R(X))$ such that $f \prec g$. Is there some $h \in U^+(R(X))$ such that

$$f \prec h^4 \prec g?$$

Thank you very much for your attention!