

GEOMETRIC CONFIGURATION OF INTEGRALLY CLOSED NOETHERIAN DOMAINS

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ABSTRACT. In this paper, we completely describe the family of integrally closed Noetherian domains between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$. We accomplish this result by classifying the Krull domains between these two polynomial rings. To this end, we first describe the DVRs of $\mathbb{Q}(X)$ lying over $\mathbb{Z}_{(p)}$ for some prime $p \in \mathbb{Z}$, by distinguishing them according to whether the extension of the residue fields is algebraic or transcendental. We unify the known descriptions of such valuations by considering ultrametric balls in $\mathbb{C}_p$, the completion of the algebraic closure of the field $\mathbb{Q}_p$ of p -adic numbers. We then study when the intersection R of such DVRs with $\mathbb{Q}[X]$ is of finite character, so that R is a Krull domain, and we finally compute the divisor class group of R . It turns out that such a ring is formed by those polynomials which simultaneously map a finite union of ultrametric balls of $\mathbb{C}_p$ to its valuation domain $\mathbb{O}_p$, as $p \in \mathbb{Z}$ ranges through the set of primes. By a result of Heinzer, the Krull domains of this class are precisely the integrally closed Noetherian domains between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$. This novel approach provides a geometric understanding of this class of integrally closed domains. Furthermore, we also describe the UFDs between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$.

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INTRODUCTION

Let $\mathbf{X}$ be a topological space, $\mathbb{C}$ be the field of complex numbers and $\mathbf{C}(\mathbf{X}, \mathbb{C})$ be the ring of continuous complex-valued functions on $\mathbf{X}$. Then $\mathbf{C}(\mathbf{X}, \mathbb{C})$ is a commutative ring with identity. It is known that $\mathbf{C}(\mathbf{X}, \mathbb{C})$ is an integral domain if and only if $\mathbf{X}$ is irreducible, i.e., $\mathbf{X}$ cannot be expressed as a finite union of proper closed subsets. It is also known that $\mathbf{C}(\mathbf{X}, \mathbb{C})$ is Noetherian if and only if $\mathbf{X}$ is a finite set, and in this case, if $\mathbf{X}$ is Hausdorff, then $\mathbf{C}(\mathbf{X}, \mathbb{C}) \cong \mathbb{C}^{|\mathbf{X}|}$ which is a zero-dimensional Noetherian ring. The class of Noetherian domains is of the utmost importance in all areas of mathematics and especially in commutative algebra and algebraic geometry.

Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ and X be an indeterminate over $\mathbb{C}$, then we have the following diagram of subrings of $\mathbf{C}(\mathbb{C}, \mathbb{C})$:

$$\begin{array}{ccccc}
 & & \overline{\mathbb{Q}}[X] & & \\
 & \nearrow & & \searrow & \\
 \mathbb{Z}[X] & \longrightarrow & \mathbb{Q}[X] & & \mathbb{C}[X] \longrightarrow \mathbf{C}(\mathbb{C}, \mathbb{C}) \\
 & \searrow & & \nearrow & \\
 & & \mathbb{R}[X] & &
 \end{array}$$

Although $\mathbf{C}(\mathbb{C}, \mathbb{C})$ is neither an integral domain nor a Noetherian ring, $\mathbf{C}(\mathbb{C}, \mathbb{C})$ contains the Noetherian integral domain $\mathbb{C}[X]$. Hence, we can ask whether we can classify the Noetherian subrings of $\mathbf{C}(\mathbb{C}, \mathbb{C})$. However, it seems to be a very difficult task, so in this paper, as the first step in a long journey towards that end, we will classify all integrally closed Noetherian subrings of $\mathbb{Q}[X]$ containing $\mathbb{Z}[X]$. For this purpose, we describe the Krull domains between these polynomial rings. We now recall the definition of this class of integral domains.

An integral domain is said to be a discrete valuation ring (DVR) if it is a principal ideal domain with exactly one nonzero prime ideal; equivalently, it is a Noetherian valuation domain that is not a field. Let D be an integral domain with quotient field K and $X^1(D)$ be the set of height one prime ideals of D . Then D is a Krull domain if the following three conditions are satisfied:

- (a) $D = \bigcap_{P \in X^1(D)} D_P$,
- (b) D_P is a DVR for all $P \in X^1(D)$, and
- (c) each nonzero nonunit of D is a unit of D_P for all but a finite number of $P \in X^1(D)$.

The class of Krull domains includes Dedekind domains, unique factorization domains (UFDs), the integral closure of a Noetherian domain, and polynomial/power series rings over a Krull domain (see, for example, [13] and [14]). In particular, if D is a field, then $X^1(D) = \emptyset$, so it satisfies the three conditions of a Krull domain. Thus, a field is a Krull domain.

Let $\mathbb{Z}$ be the ring of integers, $\mathbb{Q}$ be the field of rational numbers, and X be an indeterminate over $\mathbb{Q}$. Following [10], we say that a subring of $\mathbb{Q}[X]$ containing $\mathbb{Z}[X]$ is a polynomial overring of $\mathbb{Z}$. In [27], Peruginelli called a polynomial overring of $\mathbb{Z}$ a polynomial Dedekind domain if it is a Dedekind domain. He then characterized a polynomial Dedekind domain with the property that its residue fields of prime characteristic are finite fields, and its Picard group. Later, in [28], Peruginelli completely classified the polynomial Dedekind domains R by removing the condition on the cardinality of the residue fields; in particular, he showed that R is equal to a ring of integer-valued polynomials over an algebra. In this paper, we say that a polynomial overring of $\mathbb{Z}$ is a polynomial Krull domain if it is also a Krull domain and we are going to completely characterize the polynomial Krull domains which allows us to give a complete characterization of integrally closed Noetherian domains that are subring $\mathbb{Q}[X]$ containing $\mathbb{Z}[X]$.

It is known that D is a Krull domain if and only if there exists a family F of distinct DVRs with quotient field K such that (i) $D = \bigcap_{V \in F} V$ and (ii) every nonzero element of D is a unit in all but a finite number of rings in F [23, Proposition 3], and in this case, F is called a defining family for the Krull domain D .

Let E be the ring of entire functions. Then $E = \bigcap_{Q \in X^1(E)} E_Q$, E_Q is a DVR for all $Q \in X^1(E)$, but the intersection $E = \bigcap_{Q \in X^1(E)} E_Q$ is not of finite character, i.e., E does not satisfy the condition (c) of Krull domains, so E is not a Krull domain. Hence, in order to characterize the polynomial Krull domains, we first classify the DVRs of $\mathbb{Q}(X)$, then we have to prove when a family of such DVRs has the finite character property (i.e., property (c) above).

Let W be a DVR of $\mathbb{Q}(X)$ with maximal ideal M . Then either $W \cap \mathbb{Q} = \mathbb{Q}$ or $W \cap \mathbb{Q} = \mathbb{Z}_{(p)}$ for some prime number $p \in \mathbb{Z}$. It is well known that $W \cap \mathbb{Q} = (0)$ if and only if $W = \mathbb{Q}[X]_{f\mathbb{Q}[X]}$ for some irreducible $f \in \mathbb{Q}[X]$ or $W = \mathbb{Q}[\frac{1}{X}]_{(\frac{1}{X})}$ (see for example [11]). In this paper, we are mostly interested in the case $W \cap \mathbb{Q} = \mathbb{Z}_{(p)}$. In [28], the second-named author of this paper completely classified the DVRs (W, M) such that W/M is algebraic over

$\mathbb{Z}/p\mathbb{Z}$. In this paper, we will recall the classification of the DVRs (W, M) such that W is a residually transcendental extension of $\mathbb{Z}_{(p)}$, i.e., W/M is transcendental over $\mathbb{Z}/p\mathbb{Z}$, along the line of the work of Alexandru and Popescu ([1]). This paper consists of five sections including the introduction. In Sections 1. and 1.2, we first review the notion of the ideal factorization property of Krull domains, the divisor class group of Krull domains and rings of integer-valued polynomials, which allow us to have a nice representation for these class of polynomial Krull domains. In Section 1.3, we recall the notion of monomial valuation, which is a standard way to extend a valuation domain V with quotient field K to the field of rational functions $K(X)$ and we use it in order to describe the set $\mathcal{W}_{\text{rt}}$ of residually transcendental extensions of V to $K(X)$. In Section 1.3.2 we show how it is possible to parametrize the elements of $\mathcal{W}_{\text{rt}}$ by means of ultrametric balls in $\overline{K}$ with respect to a fixed extension U of V . It is well-known that a residually transcendental extension W of V to $K(X)$ is a torsion extension of V , i.e., the value group of W is torsion over that of V . We study in Section 2 the ordering of torsion extensions W_1, W_2 of V to $K(X)$, that is we say that $W_1 \leq W_2$ if $W_1 \cap K[X] \subseteq W_2 \cap K[X]$. If this happens, in particular we have $W_1 \cap W_2 \cap K[X] = W_1 \cap K[X]$. In Section 2.2 we study residually transcendental extension of the completion $\widehat{V}$ of V and show that $\mathcal{W}_{\text{rt}}$ is in bijection with the residually transcendental extensions of $\widehat{V}$ to $\widehat{K}(X)$.

In Section 3, we use the results of Section 2 to parametrize the residually transcendental extensions of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$ for some prime number $p \in \mathbb{Z}$ in terms of ultrametric balls in the algebraic closure $\overline{\mathbb{Q}_p}$ of the p -adic field $\mathbb{Q}_p$. We then unify the descriptions of DVRs of $\mathbb{Q}(X)$ which are either residually transcendental or residually algebraic over $\mathbb{Z}_{(p)}$ by means of ultrametric balls with possibly infinite radius in $\mathbb{C}_p$, the completion of $\overline{\mathbb{Q}_p}$. We show that there is a partial order on this set of DVRs which mirrors the containment between this ultrametric balls, modulo the action of the Galois group $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ (Theorem 3.19). Finally, in Section 4, we study first the class of Krull domains between $\mathbb{Z}_{(p)}[X]$ and $\mathbb{Q}[X]$ for a fixed prime $p \in \mathbb{Z}$ (§4.2), and as a special case we characterize the UFDs between these two polynomial rings (§4.2.3). We completely describe the prime spectrum of such a ring, characterizing when a prime ideal is maximal in geometrical terms. We then globalize these results in Section 4.3. Namely, we consider a nonempty set Λ of integer primes, and, for each $p \in \Lambda$, a finite set of DVRs $W_{p,j}$ of $\mathbb{Q}(X)$

with $X \in W_{p,j}$ and $W_{p,j} \cap \mathbb{Q} = \mathbb{Z}_{(p)}$, and $R = \bigcap_{p \in \Lambda} \left(\bigcap_{j=1}^{m_p} (W_{p,j} \cap \mathbb{Q}[X]) \right)$. We study which one of $W_{p,j}$'s is superfluous and when R is of finite character, so that R is a Krull domain (Theorem 4.34). We also compute the divisor class group $\text{Cl}(R)$ of the Krull domain R , showing that $\text{Cl}(R)$ is a direct sum of a countable family of finitely generated abelian groups and, conversely, we show that for any such a group G there exists a Krull non-Dedekind domain between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ with divisor class group isomorphic to G (§4.3.4).

Furthermore, we also characterize UFDs contained between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ (Corollary 4.37).

1. PRELIMINARIES

1.1. Algebraic properties of Krull domains. Let D be an integral domain with quotient field K and $F(D)$ be the set of nonzero fractional ideals I of D , i.e., I is a D -submodule of K such that $dI \subseteq D$ for some $0 \neq d \in D$. For each $I, J \in F(D)$, let $(I :_K J) = \{x \in K \mid xJ \subseteq I\}$, then $(I :_K J) \in F(D)$. In particular, $I^{-1} = (D :_K I)$ and $I_v = (I^{-1})^{-1}$, so $(I^{-1})_v = (I_v)^{-1} = I^{-1}$. An $I \in F(D)$ is said to be invertible if $II^{-1} = D$. It is known that I is invertible if and only if I is finitely generated and ID_M is principal for all maximal ideals M of D , and that if I is invertible, then $(II^{-1})_v = D$ and $I_v = I$.

1.1.1. Ideal factorization property of Krull domains. A Dedekind domain D is an integral domain in which each nonzero ideal can be written as a finite product of prime ideals. Then D is a Dedekind domain if and only if D is an integrally closed one-dimensional Noetherian domain, if and only if each nonzero ideal of D is invertible. It is known that D is a Dedekind domain if and only if D is a Krull domain of (Krull) dimension at most one. Furthermore, the integral closure of a Noetherian domain is a Krull domain [23, Theorem 2], so an overring of 2-dimensional Noetherian domain is a Krull domain if and only if it is an integrally closed Noetherian domain [18, Theorem 9]. Hence, a Krull domain can be considered as a higher dimensional generalization of Dedekind domains, so it is natural to ask whether a Krull domain can be characterized by an ideal factorization property as in Dedekind domains.

For $I \in F(D)$, let $I_t = \bigcup \{J_v \mid J \in F(D), J \subseteq I \text{ and } J \text{ is finitely generated}\}$. Then $I_t \in F(D)$ and the map $t : F(D) \rightarrow F(D)$, given by $I \mapsto I_t$, is well defined. The t -operation also has the following four properties for all $I, J \in F(D)$ and $0 \neq a \in K$: (i) $D_t = D$, (ii) $(aI)_t = aI_t$, (iii) $I \subseteq I_t$ and if $I \subseteq J$, then $I_t \subseteq J_t$ and (iv) $(I_t)_t = I$. An $I \in F(D)$ is said to be t -invertible if $(II^{-1})_t = D$. We then use the t -operation to characterize Krull domains as follows: D is a Krull domain if and only if, given a nonzero ideal I of D , $I_t = (P_1^{e_1} \cdots P_n^{e_n})_t$ for some $P_1, \dots, P_n \in X^1(D)$ and positive integers $e_1, \dots, e_n$, if and only if, given a nonzero element a of D , $aD = (P_1^{e_1} \cdots P_n^{e_n})_t$ for some $P_1, \dots, P_n \in X^1(D)$ and positive integers $e_1, \dots, e_n$, if and only if every nonzero (prime) ideal of D is t -invertible, if and only if every nonzero prime ideal of D contains a t -invertible prime ideal; in this case, $I_v = I_t$ for all $I \in F(D)$ (see, for example, [21]).

1.1.2. Divisor class group of a Krull domain. Let $\mathcal{D}(D) = \{I_v \mid I \in F(D)\}$. Then $\mathcal{D}(D)$ is a commutative semigroup under $I * J = (IJ)_v$, and $\mathcal{D}(D)$ is a group, i.e., $(II^{-1})_v = D$ for all $I \in \mathcal{D}(D)$, if and only if D is completely integrally closed. Next, let $\text{Inv}(D)$ be the group of invertible fractional ideals

of D and $\text{Prin}(D) = \{xD \mid x \in K, x \neq 0\}$. Then $\text{Inv}(D)$ is a subgroup of $\mathcal{D}(D)$ and $\text{Prin}(D)$ is a subgroup of $\text{Inv}(D)$, so $\text{Prin}(D) \subseteq \text{Inv}(D) \subseteq \mathcal{D}(D)$. The divisor class group of a completely integrally closed domain D is defined by the factor group $\text{Cl}(D) = \mathcal{D}(D)/\text{Prin}(D)$ of $\mathcal{D}(D)$ modulo $\text{Prin}(D)$ and the ideal class group or Picard group of an integral domain D is defined by the factor group $\text{Inv}(D)/\text{Prin}(D)$ of $\text{Inv}(D)$ modulo $\text{Prin}(D)$. Hence, if D is completely integrally closed, then $\text{Pic}(D)$ is a subgroup of $\text{Cl}(D)$.

It is known that $x \in K$ is almost integral over D if and only if $xI \subseteq I$ for some nonzero ideal I of D , so a valuation domain V is completely integrally closed if and only if the (Krull) dimension of V is at most one; in particular, a DVR is completely integrally closed. A Krull domain D is defined by an intersection of DVRs, so D is completely integrally closed and its divisor class group is also well defined.

A π -domain D is a Krull domain with $\text{Pic}(D) = \text{Cl}(D)$; equivalently, every nonzero principal ideal of D can be written as a finite product of prime ideals [21, IV. π -domains]. Hence, a Dedekind domain D is a π -domain and $\text{Pic}(D) = \text{Cl}(D)$. A UFD is a Krull domain, and Krull domain D is a UFD if and only if $\text{Cl}(D) = \{0\}$. A Krull domain D is called an almost factorial domain if $\text{Cl}(D)$ is torsion; equivalently, given $a, b \in D$, there is a positive integer n , which depends on a, b , so that $a^n D \cap b^n D$ is principal. Therefore, the divisor class group of a Krull domain D measures how far D is from a UFD and reflects the ideal factorization property of a Krull domain.

1.2. Rings of integer-valued polynomials. Let D be an integral domain with quotient field K , X be an indeterminate over D , $D[X]$ be the polynomial ring over D , and A be a torsion-free D -algebra. By embedding A and K in the extended K -algebra $B = A \otimes_D K$, we may evaluate a polynomial $f \in K[X]$ at an element $a \in A$, so that $f(a) \in B$. If $f(a) \in A$, we say that $f(X)$ is integer-valued at a . Given a subset S of A , we define the ring of integer-valued polynomials on S as

$$\text{Int}_K(S, A) = \{f \in K[X] \mid f(s) \in A, \forall s \in S\}$$

Clearly, $D[X] \subseteq \text{Int}_K(S, A) \subseteq K[X]$. If $S = A$, we set $\text{Int}_K(A) = \text{Int}_K(A, A)$, and if $A = D$, then we omit the subscript K , i.e., $\text{Int}(S, D) = \text{Int}_K(S, D)$ and $\text{Int}(D) = \text{Int}_K(D, D)$.

Given a valuation domain V with quotient field K , the *polynomial closure* of a subset S of K is defined as the largest subset $\overline{S}$ of K for which

$$\text{Int}(S, V) = \text{Int}(\overline{S}, V).$$

A subset S of K is *polynomially closed* if $S = \overline{S}$. If V has rank one, Chabert proves in [9] that the polynomially closed subsets of K form a topology of closed subsets of K . For valuation domains of higher rank, the result is no more true [30].

It is well known that $\text{Int}(\mathbb{Z})$ is a non-Noetherian two-dimensional Prüfer domain [8, Remarks VI.1.8] and $\text{Pic}(\text{Int}(\mathbb{Z}))$ is a free abelian group on a countably infinite basis [16, Corollary 7]. Let $\mathbb{P}$ be the set of prime numbers

of $\mathbb{Z}$. For $p \in \mathbb{P}$, let $\mathbb{Z}_p$ be the ring of p -adic integers, $T_p = \{\alpha \in \mathbb{Z}_p \mid \alpha \text{ is transcendental over } \mathbb{Q}\}$, $\mathbb{Z}_{(p),\alpha} = \{\varphi \in \mathbb{Q}(X) \mid \varphi(\alpha) \in \mathbb{Z}_p\}$, and $M_{(p),\alpha} = \{\varphi \in \mathbb{Q}(X) \mid \varphi(\alpha) \in p\mathbb{Z}_p\}$ for $\alpha \in T_p$. Then $\mathbb{Z}_{(p),\alpha}$ is a DVR of $\mathbb{Q}(X)$ with maximal ideal $M_{(p),\alpha}$,

$$\text{Int}(\mathbb{Z}) = \bigcap_{p \in \mathbb{P}} \left(\bigcap_{\alpha \in T_p} (\mathbb{Z}_{(p),\alpha} \cap \mathbb{Q}[X]) \right),$$

$\text{ht}(M_{(p),\alpha} \cap \text{Int}(\mathbb{Z})) = 1$, and $\mathbb{Z}_{(p),\alpha}/M_{(p),\alpha} = \mathbb{Z}/p\mathbb{Z}$. It is worthwhile to note that if $R = \text{Int}(\mathbb{Z})$, then $R = \bigcap_{P \in X^1(R)} R_P$, R_P is a DVR for all $P \in X^1(R)$, but R is not a Krull domain.

1.3. Residually transcendental extensions. Let V be a valuation domain with quotient field K , X be an indeterminate over K , and $K(X)$ be the field of rational function, i.e., the quotient field of $K[X]$. We denote by v a valuation on K with associated valuation domain V and by Γ_v the value group of v . We also denote by $\Gamma_{\bar{v}}$ the divisible hull of Γ_v , namely, $\Gamma_{\bar{v}} = \Gamma_v \otimes_{\mathbb{Z}} \mathbb{Q}$.

In this section we recall some known facts about residually transcendental extensions of V to $K(X)$, that is, valuation domains W of the field $K(X)$ lying above V (i.e., $W \cap K = V$) such that the extension of residue fields is transcendental. We denote by $\mathcal{W}_{\text{rt}}$ the set of residually transcendental extensions of V to $K(X)$. In contrast, we say that an extension W of V to $K(X)$ is residually algebraic if the extension of the residue fields is algebraic.

1.3.1. Monomial valuations. In this subsection we review the notion of monomial valuations, which is a particular class of extensions of V to $K(X)$, in order to describe the class of residually transcendental extensions.

Definition 1.1. Let $\alpha \in K$ and δ be an element of a value group Γ which contains Γ_v . For $f \in K[X]$ such that $f(X) = a_0 + a_1(X - \alpha) + \dots + a_n(X - \alpha)^n$, we set:

$$v_{\alpha,\delta}(f) = \inf\{v(a_i) + i\delta \mid i = 0, \dots, n\}$$

The function $v_{\alpha,\delta}$ naturally extends to a valuation on $K(X)$ ([7, Chapt. VI, §. 10, Lemme 1]), and it is called a *monomial valuation*. We denote by $V_{\alpha,\delta}$ the valuation domain of $K(X)$ associated to $v_{\alpha,\delta}$. Clearly, $V_{\alpha,\delta}$ lies over V . Note that $V_{0,0} = V(X)$, i.e., the case $\alpha = 0$ and $\delta = 0$ corresponds to the trivial Gaussian extension. Note also that $V_{\alpha,\delta} \cap K[X] = V[\frac{X-\alpha}{c}]$, where $c \in K$ has value δ (see for example [26, Remark 3.4]).

Let $\bar{K}$ be the algebraic closure of K , u be an extension of v to $\bar{K}$, and U be the valuation domain of u . Given $(\alpha, \delta) \in \bar{K} \times \Gamma_{\bar{v}}$, we denote by $v_{\alpha,\delta}$ the restriction of $u_{\alpha,\delta}$ to $K(X)$; note that this valuation depends on the chosen extension u of v to $\bar{K}$, see [26, Remark 3.9]. If $V_{\alpha,\delta}$ ($U_{\alpha,\delta}$, resp.) is the valuation domain associated to $v_{\alpha,\delta}$ ($u_{\alpha,\delta}$, resp.) then clearly we have $V_{\alpha,\delta} = U_{\alpha,\delta} \cap K(X)$. We also say that $(\alpha, \delta) \in \bar{K} \times \Gamma_{\bar{v}}$ is a *minimal*

pair of definition of $V_{\alpha,\delta}$ (with respect to K) if for each $\beta \in \overline{K}$ such that $u(\beta - \alpha) \geq \delta$, we have $[K(\beta) : K] \geq [K(\alpha) : K]$. Equivalently, among the set of pairs which define the valuation $u_{\alpha,\delta}$, the degree $[K(\alpha) : K]$ is minimal (see [1, Proposition 3] and [2]). We stress that when $\alpha \in \overline{K}$, the coefficients $a_0, \dots, a_n$ of the $(X - \alpha)$ -adic expansion of $f \in K[X]$ lie in $\overline{K}$ and may not be in K .

Remark 1.2. Suppose that $\delta > v(a)$ for every $a \in K$. We denote such δ by ∞ . Given $\alpha \in \overline{K}$, for every $f \in K[X]$ with $f(X) = a_0 + a_1(X - \alpha) + \dots + a_n(X - \alpha)^n$ where $a_0, \dots, a_n \in \overline{K}$ we have

$$v_{\alpha,\delta}(f) = u(a_0) = u(f(\alpha))$$

so in this case $V_{\alpha,\infty} = V_\alpha = \{\phi \in K(X) \mid \phi(\alpha) \in U\}$.

Remark 1.3. (1) Let U be a fixed extension of V to $\overline{K}$. Given $\alpha \in \overline{K}$ and an element δ in a totally ordered group containing Γ_v , it is well-known that $V_{\alpha,\delta}$ is a residually transcendental extension of V if and only if δ belongs to $\Gamma_{\overline{v}}$ (see for example [2] or [26, Lemma 3.5]). In this case, by [2, Theorem 2.1], the value group of $V_{\alpha,\delta}$ has finite index over the value group Γ_v of V . In particular, if V is a DVR, then a residually transcendental extension of V to $K(X)$ is a DVR, also.

(2) Let V be a DVR and U a fixed extension of V to $\overline{K}$. Given a (minimal) pair $(\alpha, \delta) \in \overline{K} \times \Gamma_{\overline{v}}$, the intersection $V_{\alpha,\delta} \cap K[X]$ is clearly a Krull domain which is not Prüfer, since the valuation overring $V_{\alpha,\delta}$ is residually transcendental over V , see [26, Theorem 3.15].

It is known that if we restrict to the family of Krull domains $\{V_{\alpha,\delta} \cap K[X] \mid (\alpha, \delta) \in \overline{K} \times \Gamma_{\overline{v}}\}$, this family has a natural ordering, namely:

$$(1.4) \quad V_{\alpha_1,\delta_1} \cap K[X] \subseteq V_{\alpha_2,\delta_2} \cap K[X] \Leftrightarrow \delta_1 \leq \delta_2 \text{ and } v(\alpha_1 - \alpha_2) \geq \delta_1$$

(see [4, Proposition 1.1]; see also [26, Remark 3.10]), so

$$(1.5) \quad V_{\alpha_1,\delta_1} \cap K[X] = V_{\alpha_2,\delta_2} \cap K[X] \Leftrightarrow \delta_1 = \delta_2 \text{ and } v(\alpha_1 - \alpha_2) \geq \delta_1$$

We will see in Lemma 2.1 that the previous conditions are indeed equivalent to $V_{\alpha_1,\delta_1} = V_{\alpha_2,\delta_2}$.

If $(\alpha, \delta) \in \overline{K} \times \Gamma_{\overline{v}}$ and we consider the ball $B(\alpha, \delta) = \{x \in \overline{K} \mid v(x - \alpha) \geq \delta\}$, then condition (1.4) holds if and only if $B(\alpha_2, \delta_2) \subseteq B(\alpha_1, \delta_1)$. In the case $V = \mathbb{Z}_{(p)}$ for some prime p , we generalize this ordering to the case of pairs $(\alpha, \delta) \in \mathbb{C}_p \times \mathbb{Q}$ in Section 3.2.

The main result of Alexandru, Popescu and Zaharescu is the following characterization of the residually transcendental extensions of V to $K(X)$ (see [1, 2, 4]).

Theorem 1.6. *Let V be a valuation domain with quotient field K and let U be a fixed extension of V to $\overline{K}$. Let W be a residually transcendental extension of V to $K(X)$. Then there exists a minimal pair $(\alpha, \delta) \in \overline{K} \times \Gamma_{\overline{v}}$ such that W is the restriction to $K(X)$ of the monomial valuation $U_{\alpha,\delta}$ of the field $\overline{K}(X)$.*

Note that the valuation domain $U_{\alpha,\delta}$ of $\overline{K}(X)$ in Theorem 1.6 is a residually transcendental extension of U to $\overline{K}(X)$. The next lemma is a generalization of [26, Lemma 3.13] to valuation domains of arbitrary rank.

Lemma 1.7. *Let U be an extension of V to $\overline{K}$ and let $(\alpha, \delta) \in \overline{K} \times \Gamma_v$. Then*

$$V[X] \subset U_{\alpha,\delta} \cap K(X) \Leftrightarrow u(\alpha) \geq 0 \text{ and } \delta \geq 0.$$

Proof. Note that the containment is equivalent to $u_{\alpha,\delta}(X) \geq 0$ (and, in particular, equivalent to $U[X] \subset U_{\alpha,\delta}$). We have $u_{\alpha,\delta}(X) = \min\{\delta, u(\alpha)\}$ so that the claim is true. $\square$

1.3.2. Parametrization of residually transcendental extensions. In this subsection, we show how to parametrize $\mathcal{W}_{\text{rt}}$ by means of ultrametric balls contained in $\overline{K}$ with respect to some fixed extension U of V to $\overline{K}$.

Let G_K be the absolute Galois group $\text{Gal}(\overline{K}/K)$. Note that G_K is also equal to the Galois group of $\overline{K}(X)/K(X)$. Given an extension U of V to $\overline{K}$, we denote by $\mathcal{D}(U)$ the decomposition group of U , namely: $\mathcal{D}(U) = \{\sigma \in G_K \mid \sigma(U) = U\}$.

Lemma 1.8. *Let U be a fixed extension of V to $\overline{K}$. Let $(\alpha, \delta) \in \overline{K} \times \Gamma_{\overline{v}}$, with $\delta = u(c)$ for some $c \in \overline{K}$, and $\sigma \in G_K$. Then we have*

$$\sigma(U_{\alpha,\delta}) = \sigma(U)_{\sigma(\alpha),\delta}.$$

In particular, if $\sigma \in \mathcal{D}(U)$, then $\sigma(U_{\alpha,\delta}) = U_{\sigma(\alpha),\delta}$.

Proof. Since $\frac{X-\alpha}{c}$ is a unit in $U_{\alpha,\delta}$ and $U_{\alpha,\delta}$ is residually transcendental over the residue field k_U of U , it follows that $\sigma\left(\frac{X-\alpha}{c}\right) = \frac{X-\sigma(\alpha)}{\sigma(c)}$ is a unit in $\sigma(U_{\alpha,\delta})$ and $\sigma(U_{\alpha,\delta})$ is residually transcendental over $k_{\sigma(U)}$ (note that $\sigma(U_{\alpha,\delta})$ is an extension of $\sigma(U)$ to $\overline{K}(X)$). Recall that the valuation associated to $\sigma(U) = U'$ is equal to $u' = u \circ \sigma^{-1}$ ([12, Proposition 3.2.16, (1)]), so $u'(\sigma(c)) = \delta$. Then the claim follows by [7, Chapt. VI, §10, Proposition 2].

If $\sigma \in \mathcal{D}(U)$, then the last claim follows at once. $\square$

We give next a criterion for when two residually transcendental extensions of U to $\overline{K}(X)$ contracts down to the same valuation domain of $K(X)$ (which is a residually transcendental extension of V to $K(X)$). This has been done in a similar way in [24]. In particular, given a residually transcendental extension $V_{\alpha,\delta}$ of V to $K(X)$, where $(\alpha, \delta) \in \overline{K} \times \Gamma_{\overline{v}}$, we describe the set of extensions of $V_{\alpha,\delta}$ to $\overline{K}(X)$ lying above U (see also [3, Theorem 2.2]).

Theorem 1.9. *Let U be a fixed extension of V to $\overline{K}$ and let $(\alpha, \delta), (\alpha', \delta') \in \overline{K} \times \Gamma_{\overline{v}}$. Then the following conditions are equivalent.*

- (1) $U_{\alpha,\delta} \cap K(X) = U_{\alpha',\delta'} \cap K(X)$.
- (2) $\sigma(U_{\alpha,\delta}) = U_{\alpha',\delta'}$ for some $\sigma \in \mathcal{D}(U)$.
- (3) $\delta = \delta'$ and $u(\sigma(\alpha) - \alpha') \geq \delta$, for some $\sigma \in \mathcal{D}(U)$.

In particular, the set of extensions of a residually transcendental extension $W = U_{\alpha,\delta} \cap K(X)$ to $\overline{K}(X)$ lying above U is equal to $\{U_{\sigma(\alpha),\delta} \mid \sigma \in \mathcal{D}(U)\}$.

Proof. (1) $\Rightarrow$ (2) By [12, Conjugation Theorem 3.2.15], there exists $\sigma \in G_K$ such that $\sigma(U_{\alpha,\delta}) = U_{\alpha',\delta'}$. We claim that $\sigma \in \mathcal{D}(U)$. In fact, let $x \in U \subset U_{\alpha,\delta}$. Then $\sigma(x) \in \sigma(U_{\alpha,\delta}) \cap \overline{K} = U_{\alpha',\delta'} \cap \overline{K} = U$. Thus, (2) holds.

(2) $\Rightarrow$ (3) By Lemma 1.8, $U_{\sigma(\alpha),\delta} = \sigma(U_{\alpha,\delta}) = U_{\alpha',\delta'}$. By [4, Proposition 1.1], $\delta' = \delta$ and $u(\sigma(\alpha) - \alpha') \geq \delta$, which is condition (3).

(3) $\Rightarrow$ (1) The inequality $u(\sigma(\alpha) - \alpha') \geq \delta$ is equivalent to $\alpha' \in B_u(\sigma(\alpha), \delta) = \{x \in \overline{K} \mid u(\sigma(\alpha) - x) \geq \delta\}$, so that (α', δ) is another pair of definition for $U_{\sigma(\alpha),\delta}$ by [1, Proposition 3] and thus $U_{\alpha',\delta} = U_{\sigma(\alpha),\delta}$. By Lemma 1.8, we also have $U_{\sigma(\alpha),\delta} = \sigma(U_{\alpha,\delta})$, so $U_{\alpha,\delta}$ and $U_{\alpha',\delta}$ contracts down to $K(X)$ to the same valuation domain. Thus, (1) holds.

The last claim follows immediately from the previous part and Lemma 1.8. $\square$

For the next theorem, we fix the following notation: given an extension U of V to $\overline{K}$, let $\mathcal{B}_u$ be the set of ultrametric balls $B_u(\alpha, \delta) = \{x \in \overline{K} \mid u(x - \alpha) \geq \delta\}$ in $\overline{K}$ with respect to U , where $(\alpha, \delta) \in \overline{K} \times \Gamma_{\overline{v}}$. It is worthwhile to note that condition (3) of Theorem 1.9 is equivalent to the following: $\sigma(B_u(\alpha, \delta)) = B_u(\sigma(\alpha), \delta) = B_u(\alpha', \delta)$ for some $\sigma \in \mathcal{D}(U)$.

By Lemma 1.8, $\mathcal{D}(U)$ acts on $\mathcal{B}_u$, since for $\sigma \in \mathcal{D}(U)$ we have $\sigma(B_u(\alpha, \delta)) = B_u(\sigma(\alpha), \delta)$. In particular, $\mathcal{D}(U)$ does not change the radius of an ultrametric ball. We denote by $\mathcal{B}_u/\mathcal{D}(U)$ the set of equivalence classes of $\mathcal{B}_u$ under the action of the decomposition group $\mathcal{D}(U)$. By Lemma 1.8, an element of $\mathcal{B}_u/\mathcal{D}(U)$ is equal to $[B_u(\alpha, \delta)] = \{B_u(\sigma(\alpha), \delta) \mid \sigma \in \mathcal{D}(U)\}$; in particular, note that such an equivalence class is formed by finitely many closed balls (i.e., at most as many as the number of conjugates of α over K).

Theorem 1.10. *Let U be a fixed extension of V to $\overline{K}$. Then the map $\Theta : \mathcal{B}_u/\mathcal{D}(U) \rightarrow \mathcal{W}_{rt}$ given by $[B_u(\alpha, \delta)] \mapsto U_{\alpha,\delta} \cap K(X)$ is bijective.*

Proof. In order to show that the map Θ is well-defined, we observe first that given $(\alpha, \delta) \in \overline{K} \times \Gamma_{\overline{v}}$, we have $U_{\alpha,\delta} \cap K(X) \in \mathcal{W}_{rt}$ by Remark 1.3. Moreover, if $\sigma(B_u(\alpha, \delta)) = B_u(\sigma(\alpha), \delta) \in [B_u(\alpha, \delta)]$ for some $\sigma \in \mathcal{D}(U)$, then $U_{\alpha,\delta} \cap K(X) = U_{\sigma(\alpha),\delta} \cap K(X)$ by Theorem 1.9.

The surjectivity of Θ follows by Theorem 1.6. Finally, the injectivity follows again by Theorem 1.9. In fact, suppose that $\Theta([B_u(\alpha, \delta)]) = \Theta([B_u(\alpha', \delta')])$, that is, $U_{\alpha,\delta} \cap K(X) = U_{\alpha',\delta'} \cap K(X)$. Then, by Theorem 1.9, $\delta = \delta'$ and $u(\sigma(\alpha) - \alpha') \geq \delta$, so that $B_u(\alpha', \delta') = B_u(\sigma(\alpha), \delta)$. Hence, $[B_u(\alpha, \delta)] = [B_u(\alpha', \delta')]$. $\square$

2. POLYNOMIAL ORDERING OF TORSION EXTENSIONS

We say that an extension W of V to $K(X)$ is torsion if $\Gamma_w \subseteq \Gamma_{\overline{v}}$; we denote by $\mathcal{W}_t$ the set of all torsion extensions of V to $K(X)$. Note that a residually transcendental extension of V to $K(X)$ is torsion by Remark 1.3,

so $\mathcal{W}_{\text{rt}} \subseteq \mathcal{W}_t$. We stress that not all the residually algebraic extensions of V to $K(X)$ are torsion. For example, $\{\phi \in K(X) \mid \phi(0) \in V\}$ is a residually algebraic extension of V (its residue field is the same as the residue field of V) which is not torsion (its one-dimensional valuation overring is $K[X]_{(X)}$; see Proposition 4.17 and also Remark 3.3).

2.1. A partial order on the set of torsion extensions of V to $K(X)$. Let W_1, W_2 be in $\mathcal{W}_t$. In [4, page 282], the following definition is given only for residually transcendental extensions (which are torsion):

$$w_1 \leq w_2 \Leftrightarrow w_1(f) \leq w_2(f), \forall f \in K[X]$$

where the last inequality is considered in $\Gamma_{\bar{v}}$. More generally, we give the same definition for torsion extensions. We remark that since any $f \in K[X]$ can be written as $\frac{g(X)}{c}$ for some $g \in V[X]$ and $c \in V$, $c \neq 0$, and the valuations w_1, w_2 coincide on K , the previous condition is equivalent to: $w_1(g) \leq w_2(g)$ for each $g \in V[X]$. Note that clearly we have

$$w_1 = w_2 \Leftrightarrow w_1(f) = w_2(f), \forall f \in K[X] \Leftrightarrow W_1 = W_2.$$

In particular, we also say that $w_1 < w_2$ if and only if $w_1 \leq w_2$ and $w_1 \neq w_2$.

Lemma 2.1. *Let $W_i \in \mathcal{W}_t$ and M_i be the maximal ideal of W_i for $i = 1, 2$.*

- (1) *The following statements are equivalent.*
 - (a) $w_1 \leq w_2$.
 - (b) $W_1 \cap K[X] \subseteq W_2 \cap K[X]$.
 - (c) $M_1 \cap K[X] \subseteq M_2 \cap K[X]$.
- (2) $W_1 = W_2$ *if and only if* $W_1 \cap K[X] = W_2 \cap K[X]$, *if and only if* $M_1 \cap K[X] = M_2 \cap K[X]$.
- (3) $(\mathcal{W}_t, \leq)$ *is a partially ordered set.*

Proof. (1) (a) $\Rightarrow$ (c). Let $f \in M_1 \cap K[X]$ be nonzero. Then $0 < w_1(f) \leq w_2(f)$, so $f \in M_2 \cap K[X]$. Thus $M_1 \cap K[X] \subseteq M_2 \cap K[X]$.

(c) $\Rightarrow$ (b). Let $g \in W_1 \cap K[X]$ be nonzero. If $g \in M_1$, then $g \in M_2 \cap K[X] \subseteq W_2 \cap K[X]$. Next assume that $g \notin M_1$ and $g \notin W_2 \cap K[X]$. Since W_2 is a torsion extension of V , there are $n \in \mathbb{N}$ and $b \in V$ such that $w_2(g^n) = nw_2(g) = -v(b) < 0$ or $w_2(g^n b) = 0$. Note that $w_1(g^n b) = nw_1(g) + w_1(b) = v(b) > 0$, so $g^n b \in M_1 \cap K[X] \subseteq M_2 \cap K[X]$ by assumption. Hence $w_2(g^n b) > 0$, a contradiction, which implies $g \in W_2 \cap K[X]$. Thus $W_1 \cap K[X] \subseteq W_2 \cap K[X]$.

(b) $\Rightarrow$ (a). Assume that $f \in K[X]$ is nonzero. Then either $f \in W_1$ or $\frac{1}{f} \in W_1$, and since W_1 is torsion over V , there exist a positive integer $n \in \mathbb{N}$ and $0 \neq c \in K$ such that $nw_1(f) = v(c)$. Hence, $w_1(\frac{f^n}{c}) = 0$. In particular, $\frac{f^n}{c} \in W_1 \cap K[X] \subseteq W_2 \cap K[X]$, so that we also have

$$0 \leq w_2\left(\frac{f^n}{c}\right) \Leftrightarrow w_2(f) \geq \frac{v(c)}{n} = w_1(f),$$

so the inequality holds true.

(2) It is clear that $W_1 = W_2$ implies both of $W_1 \cap K[X] = W_2 \cap K[X]$ and $M_1 \cap K[X] = M_2 \cap K[X]$. Conversely, if $W_1 \cap K[X] = W_2 \cap K[X]$ or $M_1 \cap K[X] = M_2 \cap K[X]$, then $w_1(f) = w_2(f)$ for every $f \in K[X]$ by (1), which implies that $w_1 = w_2$ on $K(X)$. Thus, $W_1 = W_2$.

(3) This follows directly from (1) and (2) above. $\square$

Let the notation be as in Lemma 2.1. Then $w_1 < w_2$ if and only if $W_1 \cap K[X] \subset W_2 \cap K[X]$, if and only if $M_1 \cap K[X] \subset M_2 \cap K[X]$. The next corollary gives an alternative characterization of this strict inequality.

Corollary 2.2. *Let $W_i \in \mathcal{W}_t$ and M_i be the maximal ideal of W_i for $i = 1, 2$. Then the following statements are satisfied.*

- (1) $w_1 < w_2$ if and only if $M_1 \cap K[X] \subset M_2 \cap (W_1 \cap K[X])$, and in this case, $M_1 \cap K[X]$ is a prime non-maximal ideal of $W_1 \cap K[X]$.
- (2) $W \in \mathcal{W}_t$ is a maximal element of $(\mathcal{W}_t, \leq)$ if and only if W is residually algebraic.

Proof. (1) Assume that $w_1 < w_2$. Then, by Lemma 2.1, there exists $f \in (W_2 \cap K[X]) \setminus (W_1 \cap K[X])$, so $w_1(f) < 0 \leq w_2(f)$. Since W_1 is a torsion extension of V , there exist $n \in \mathbb{N}$ and $0 \neq a \in K$ such that $w_1(f^n) = v(a)$. Then, we have

$$0 = w_1\left(\frac{f^n}{a}\right) < w_2\left(\frac{f^n}{a}\right),$$

so $\frac{f^n}{a} \in M_2 \cap (W_1 \cap K[X])$ but $\frac{f^n}{a} \notin M_1 \cap K[X]$. Conversely, suppose that $M_1 \cap K[X] \subset M_2 \cap (W_1 \cap K[X])$. Then $M_2 \cap (W_1 \cap K[X]) \subseteq M_2 \cap K[X]$ implies $M_1 \cap K[X] \subset M_2 \cap K[X]$. Thus $w_1 < w_2$ by Lemma 2.1.

(2) $(\Rightarrow)$ Suppose that W is residually transcendental. By Theorem 1.6, there exists $(\alpha, \delta) \in \overline{K} \times \Gamma_{\overline{v}}$ such that $W = U_{\alpha, \delta} \cap K(X)$ (where U is a fixed extension of V to $\overline{K}$). If $\delta' \in \Gamma_v$ is strictly greater than δ , we have $W \cap K[X] \subsetneq W' \cap K[X]$, where $W' = U_{\alpha, \delta'} \cap K(X)$. In fact, $U_{\alpha, \delta} \cap \overline{K}[X] \subsetneq U_{\alpha, \delta'} \cap \overline{K}[X]$ because $\frac{X - \alpha}{c'}$ is in the latter ring but not in the former (where $c' \in K$ has value δ') and by Theorem 1.9, $U_{\alpha, \delta} \cap K[X] \subsetneq U_{\alpha, \delta'} \cap K[X]$.

$(\Leftarrow)$ Suppose that $W \in \mathcal{W}_t$ is residually algebraic. We have the following containments, where M is the maximal ideal of W :

$$\frac{V}{M \cap K} \subseteq \frac{W \cap K[X]}{M \cap K[X]} \subseteq \frac{W}{M}.$$

It follows that $(W \cap K[X])/(M \cap K[X])$ is a field and thus $M \cap K[X]$ is a maximal ideal of $W \cap K[X]$. By the first part of the Corollary, W is a maximal element of $(\mathcal{W}_t, \leq)$. $\square$

It is worthwhile to note that if K is algebraically closed and maximally complete (i.e., it does not admit any proper immediate extension), then it is straightforward to see that there are no residually algebraic torsion extensions of V to $K(X)$. Hence, in this case, $\mathcal{W}_t$ has no maximal element by Corollary 2.2.

2.2. Residually transcendental extensions over the completion. Let $\widehat{V}$ (resp. $\widehat{K}$) denote the v -adic completion of V (resp. K), $\widetilde{\mathcal{W}}_t$ be the set of all torsion extensions of $\widehat{V}$ to $\widehat{K}(X)$, and $\widetilde{\mathcal{W}}_{rt}$ be the set of all residually transcendental extensions of $\widehat{V}$ to $\widehat{K}(X)$. In this subsection, we show that the map from $\widetilde{\mathcal{W}}_{rt}$ into $\mathcal{W}_{rt}$ given by $W \mapsto W \cap K(X)$ is bijective.

Lemma 2.3. *Let $\widetilde{W}$ be an extension of $\widehat{V}$ to $\widehat{K}(X)$ such that $\tilde{w}(X) > v(a)$, for some $a \in K$. If $W = \widetilde{W} \cap K(X)$, then we have*

$$\frac{\widetilde{W} \cap \widehat{K}[X]}{\widetilde{M} \cap \widehat{K}[X]} = \frac{W \cap K[X]}{M \cap K[X]},$$

where $\widetilde{M}, M$ are the maximal ideals of $\widetilde{W}, W$, respectively.

Proof. Clearly, we have the containment ($\supseteq$). For the other containment, we need to show that for every $f \in \widetilde{W} \cap \widehat{K}[X]$ there exists $g \in W \cap K[X]$ such that $\tilde{w}(f - g) > 0$. Suppose that $f(X) = \sum_{i=0}^d \alpha_i X^i$. We choose $g \in K[X]$,

$g(X) = \sum_{i \geq 0}^d a_i X^i$, such that $v(\alpha_i - a_i) + i\tilde{w}(X) > 0$, for each $i = 0, \dots, d$. In particular,

$$(2.4) \quad \tilde{w}(f - g) \geq \min\{v(\alpha_i - a_i) + i\tilde{w}(X)\} > \min\{v(\alpha_i - a_i) + i\tilde{w}(X)\} > 0$$

so that $f - g \in \widetilde{M}$. Note that $g \in W$, so we get the claim. $\square$

Let $\widehat{V}_\infty = \{\psi \in \widehat{K}(X) \mid \psi(\infty) \in \widehat{V}\}$, where $\psi(\infty)$ is defined as $\varphi(0)$ with $\varphi(X) = \psi(1/X)$ (see [25]). Then the assumption on $\widetilde{W}$ of Lemma 2.3 is equivalent to say $\widetilde{W} \neq \widehat{V}_\infty$. Note that every $W \in \mathcal{W}_t$ satisfies the assumption of Lemma 2.3.

Lemma 2.5. *Let $\widetilde{R}$ be an integral domain such that $\widehat{V}[X/a] \subseteq \widetilde{R} \subseteq \widehat{K}[X]$ for some nonzero $a \in K$ and let $R = \widetilde{R} \cap K[X]$. Let $\widetilde{W}$ be a valuation domain of $\widehat{K}(X)$ extending $\widehat{V}$ and let $W = \widetilde{W} \cap K(X)$. Then $R \subset W$ if and only if $\widetilde{R} \subset \widetilde{W}$.*

Proof. The implication ($\Leftarrow$) is clear. Conversely, suppose that $R \subset W$ and let $f \in \widetilde{R}$. In particular, $f \in \widehat{K}[X] = \widehat{K}[X/a]$, so $f(X) = \sum_{i=0}^d \alpha_i (X/a)^i$, for some $\alpha_i \in \widehat{K}$. Let $g(X) = \sum_{i=0}^d a_i (X/a)^i \in K[X]$ be such that $v(\alpha_i - a_i) \geq 0$ for each $i = 0, \dots, d$. Then $h := f - g \in \widehat{V}[X/a] \subseteq \widetilde{R}$ so that $g = f - h \in \widetilde{R} \cap K[X] = R$. By assumption, $g \in W$ and note that $h \in \widetilde{W}$, because $V[X/a] \subset W$ implies that $\widehat{V}[X/a] \subset \widetilde{W}$. Hence, $f = h + g \in \widetilde{W}$, as claimed. $\square$

The following corollary follows at once.

Corollary 2.6. *Let $\widetilde{W}_i$ be an extension of $\widehat{V}$ to $\widehat{K}(X)$ such that $\tilde{w}_i(X) > v(a)$ for some $a \in K$, for $i = 1, 2$. Let $W_i = \widetilde{W}_i \cap K(X)$, for $i = 1, 2$. Then we have*

$$(2.7) \quad W_1 \cap K[X] \subseteq W_2 \cap K[X] \Leftrightarrow \widetilde{W}_1 \cap \widehat{K}[X] \subseteq \widetilde{W}_2 \cap \widehat{K}[X].$$

Proof. It is sufficient to notice that the assumption of $\tilde{w}_1$ implies that $\widehat{V}[X/a] \subseteq \widetilde{W}_1 \cap \widehat{K}[X]$ and then apply Lemma 2.5. $\square$

Lemma 2.8. *Let W be a residually transcendental extension of V to $K(X)$ and $\widehat{K}(X)$ (resp. $\widehat{W}$) be the completion of $K(X)$ (resp. W) with respect to W . Then the following statements hold true:*

- (1) $\widehat{K}(X)$ is a subfield of $\widehat{K(X)}$.
- (2) $\widehat{W}$ is immediate over $\widehat{W} \cap \widehat{K}(X)$.
- (3) $\widehat{W} \cap \widehat{K}(X)$ is a residually transcendental extension of $\widehat{V}$.

Proof. Since Γ_w/Γ_v is torsion, it follows that $\widehat{K}$ is equal to the topological closure of K in $\widehat{K(X)}$. Hence, we may embed $\widehat{K}(X)$ into $\widehat{K(X)}$.

Let $\widetilde{W} = \widehat{W} \cap \widehat{K}(X)$. Since $V \subseteq \widehat{V}$ and $W \subseteq \widehat{W}$ are immediate extensions, it follows that $\widehat{W}$ is immediate over $\widehat{W}$ and $\widehat{W}$ is residually transcendental over $\widehat{V}$. In particular, since $K(X) \subseteq \widehat{K}(X) \subseteq \widehat{K(X)}$, it follows that also $W \subseteq \widetilde{W}$ is an immediate extension and so $\widetilde{W}$ is also a residually transcendental extension of $\widehat{V}$. If X were algebraic over $\widehat{K}$, then $\widehat{K} \subseteq \widehat{K}(X)$ would be a finite extension and so $\widetilde{W}$ could not be residually transcendental over $\widehat{V}$. Then X is transcendental over $\widehat{K}$. $\square$

Lemma 2.9. *Let $\widetilde{W}$ be a residually transcendental extension of $\widehat{V}$ to $\widehat{K}(X)$. Then $\widetilde{W} \cap K(X)$ is a residually transcendental extension of V .*

Proof. By [1, Proposition 1], there exists $f \in \widehat{K}[X] \cap \widetilde{W}$ such that its residue modulo $\widehat{M}$ is transcendental over $\widehat{V}/\widehat{M} = V/M$. By Lemma 2.3, if $g \in K[X]$ is a polynomial of the same degree as f sufficiently v -adically close to f , then $g \in \widetilde{W} \cap K(X)$ and f and g have the same residue modulo $\widehat{M}$ so g is residually transcendental over V/M . Thus, $\widetilde{W} \cap K(X)$ is a residually transcendental extension of V . $\square$

Theorem 2.10. *If $\widetilde{W} \in \widetilde{\mathcal{W}}_t$ then $\widetilde{W} \cap K(X) \in \mathcal{W}_t$. Moreover, if $\widetilde{W}_i \in \widetilde{\mathcal{W}}_t$ and $W_i = \widetilde{W}_i \cap K(X)$ for $i = 1, 2$, then $w_1 \leq w_2$ if and only if $\tilde{w}_1 \leq \tilde{w}_2$. In particular, $W_1 = W_2$ if and only if $\widetilde{W}_1 = \widetilde{W}_2$.*

Proof. It is clear that if $\widetilde{W} \in \widetilde{\mathcal{W}}_t$, then $\widetilde{W} \cap K(X) \in \mathcal{W}_t$, because $\Gamma_v = \Gamma_{\widehat{v}}$. Hence, by Lemma 2.1, the claim $w_1 \leq w_2 \Leftrightarrow \tilde{w}_1 \leq \tilde{w}_2$ is equivalent to (2.7), which holds by Corollary 2.6.

For the final claim, ($\Leftarrow$) is straightforward. Conversely, suppose $W_1 = W_2$. Then $W_1 \cap K[X] = W_2 \cap K[X]$, so by (2.7), $\widetilde{W}_1 \cap \widehat{K}[X] = \widetilde{W}_2 \cap \widehat{K}[X]$. Thus, by Lemma 2.1, $\widetilde{W}_1 = \widetilde{W}_2$. $\square$

Corollary 2.11. *The map $\Phi : \widetilde{\mathcal{W}}_{\text{rt}} \rightarrow \mathcal{W}_{\text{rt}}$ given by $\widetilde{W} \mapsto \widetilde{W} \cap K(X)$ is bijective.*

Proof. Note that if $\widetilde{W} \in \widetilde{\mathcal{W}}_{\text{rt}}$, then $W = \widetilde{W} \cap K(X) \in \mathcal{W}_{\text{rt}}$ by Lemma 2.9. Hence, Φ is well-defined. Then Φ is surjective by Lemma 2.8 and Φ is injective by Theorem 2.10. $\square$

Remark 2.12. Let Ψ be the map from $\widetilde{\mathcal{W}}_{\text{t}}$ to $\mathcal{W}_{\text{t}}$ which sends $\widetilde{W}$ to $\widetilde{W} \cap K(X)$. Clearly, the restriction of Ψ to $\widetilde{\mathcal{W}}_{\text{rt}}$ is equal to Φ . As in the proof of Corollary 2.11, Ψ is well-defined and injective by Theorem 2.10. However, Ψ is not surjective: if $\alpha \in \widehat{K}$ is transcendental over K , then $V_\alpha = \{\phi \in K(X) \mid \phi(\alpha) \in \widehat{V}\}$ is a torsion extension of V , while $\widehat{V}_\alpha = \{\psi \in \widehat{K}(X) \mid \psi(\alpha) \in \widehat{V}\}$ is not, since $\widehat{V}_\alpha$ is contained in the DVR $\widehat{K}[X]_{(X-\alpha)}$.

3. COMPLETE LIST OF DVRs OF $\mathbb{Q}(X)$

In this section we give a complete description of the family of DVRs with quotient field $\mathbb{Q}(X)$.

Let $\mathbb{P}$ be the set of prime numbers in $\mathbb{Z}$, so $\{p\mathbb{Z} \mid p \in \mathbb{P}\}$ is the set of nonzero (principal) prime ideals of the ring $\mathbb{Z}$ of integers. Let $W \subset \mathbb{Q}(X)$ be a DVR with maximal ideal M . We have one of the following two cases:

- $W \cap \mathbb{Q} = \mathbb{Q}$.
- $W \cap \mathbb{Q} = \mathbb{Z}_{(p)}$, for some $p \in \mathbb{P}$.

In the first case we say that W is a *non-unitary* valuation domain, and in the second case, that W is a *unitary* valuation domain. For each $p \in \mathbb{P}$, let $\mathcal{W}_p$ be the set of DVRs W of $\mathbb{Q}(X)$ such that $W \cap \mathbb{Q} = \mathbb{Z}_{(p)}$.

It is a classical result (see for example [11, p. 3]) that if W is a non-unitary valuation domain, then W is equal to one of the following DVRs, according to whether $X \in W$ or $X \notin W$:

- $W = \mathbb{Q}[X]_{(g)}$, for some irreducible $g \in \mathbb{Q}[X]$.
- $W = \mathbb{Q}[1/X]_{(1/X)}$.

If $W \in \mathcal{W}_p$ for some $p \in \mathbb{P}$, we distinguish between these two subcases:

- (1) W is residually algebraic over $\mathbb{Z}_{(p)}$.
- (2) W is residually transcendental over $\mathbb{Z}_{(p)}$.

Throughout this paper, for each $p \in \mathbb{P}$ we set:

- $\mathcal{W}_{p,\text{ra}} = \{W \in \mathcal{W}_p \mid W \text{ is residually algebraic over } \mathbb{Z}_{(p)}\}$;
- $\mathcal{W}_{p,\text{rt}} = \{W \in \mathcal{W}_p \mid W \text{ is residually transcendental over } \mathbb{Z}_{(p)}\}$.

In particular, $\mathcal{W}_p = \mathcal{W}_{p,\text{ra}} \cup \mathcal{W}_{p,\text{rt}}$ and $\mathcal{W}_{p,\text{ra}} \cap \mathcal{W}_{p,\text{rt}} = \emptyset$. Moreover, let

$$\mathcal{W} = \bigcup_{p \in \mathbb{P}} \mathcal{W}_p, \quad \mathcal{W}_{\text{ra}} = \bigcup_{p \in \mathbb{P}} \mathcal{W}_{p,\text{ra}}, \quad \mathcal{W}_{\text{rt}} = \bigcup_{p \in \mathbb{P}} \mathcal{W}_{p,\text{rt}},$$

so $\mathcal{W}$ is the set of DVRs of $\mathbb{Q}(X)$, $\mathcal{W}_{\text{ra}}$ is the set of DVRs of $\mathbb{Q}(X)$ that are residually algebraic extensions of $\mathbb{Z}_{(p)}$ for some $p \in \mathbb{P}$, and $\mathcal{W}_{\text{rt}}$ is the set of DVRs of $\mathbb{Q}(X)$ that are residually transcendental extensions of $\mathbb{Z}_{(p)}$ for some $p \in \mathbb{P}$.

3.1. Extensions of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$. Let $\mathbb{Q}_p, \mathbb{Z}_p$ be the field of p -adic numbers and the ring of p -adic integers, respectively, which are the completion of $\mathbb{Q}, \mathbb{Z}_{(p)}$ with respect to the topology induced by the p -adic valuation. Let $\overline{\mathbb{Q}_p}$ be a fixed algebraic closure of $\mathbb{Q}_p$ and $\overline{\mathbb{Z}_p}$ be the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}_p}$; since $\mathbb{Q}_p$ is Henselian, $\overline{\mathbb{Z}_p}$ is a one-dimensional non-discrete valuation domain. We denote by v_p the unique extension of the p -adic valuation to $\overline{\mathbb{Q}_p}$. The ring $\overline{\mathbb{Z}_p}$ is the valuation domain of v_p (i.e., $\overline{\mathbb{Z}_p} = \{x \in \overline{\mathbb{Q}_p} \mid v_p(x) \geq 0\}$). Let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}_p}$ with respect to the topology induced by the valuation v_p and $\mathbb{O}_p$ the completion of $\overline{\mathbb{Z}_p}$. It is well-known that $\mathbb{C}_p$ is algebraically closed ([17, Proposition 5.7.8]). Clearly, each element $\alpha \in \mathbb{C}_p \setminus \overline{\mathbb{Q}_p}$ is transcendental over $\mathbb{Q}_p$, and so, in particular, such an α is transcendental over $\mathbb{Q}$; on the contrary, an element $\alpha \in \overline{\mathbb{Q}_p}$ can be either algebraic or transcendental over $\mathbb{Q}$. We denote by $\mathbb{C}_p^{\text{br}}$ the subset of those $\alpha \in \mathbb{C}_p$ such that the ramification index of the extension $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ is finite. We say that an element $\alpha \in \mathbb{C}_p^{\text{br}}$ is unramified over $\mathbb{Q}_p$ if the ramification index of $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ is 1 (we stress that such an element may not be algebraic over $\mathbb{Q}_p$). Note that $\overline{\mathbb{Q}_p} \subset \mathbb{C}_p^{\text{br}} \subset \mathbb{C}_p$. We also set $\mathbb{O}_p^{\text{br}} = \mathbb{O}_p \cap \mathbb{C}_p^{\text{br}}$.

3.1.1. Residually algebraic extensions. The following theorem characterizes the DVRs of $\mathbb{Q}(X)$ which are residually algebraic extensions of $\mathbb{Z}_{(p)}$, for some $p \in \mathbb{P}$ (see [28, Corollary 2.28]).

Theorem 3.1. *Let W be a DVR of $\mathbb{Q}(X)$ which is a residually algebraic extension of $\mathbb{Z}_{(p)}$ for some $p \in \mathbb{P}$. Then there exists $\alpha \in \mathbb{C}_p^{\text{br}}$, transcendental over $\mathbb{Q}$, such that*

$$(3.2) \quad W = \mathbb{Z}_{(p),\alpha} = \{\phi \in \mathbb{Q}(X) \mid \phi(\alpha) \in \mathbb{O}_p\}.$$

In particular, $\mathcal{W}_{p,ra} = \{\mathbb{Z}_{(p),\alpha} \mid \alpha \in \mathbb{C}_p^{\text{br}}, \alpha \text{ transcendental over } \mathbb{Q}\}$. Furthermore, $\alpha \in \overline{\mathbb{Q}_p}$ if and only if the extension of the residue fields $W/M \supseteq \mathbb{Z}/p\mathbb{Z}$ is finite.

Remark 3.3. We remark that if $\alpha \in \overline{\mathbb{Q}_p}$, we may define in general the following valuation domain as above:

$$\mathbb{Z}_{(p),\alpha} = \{\phi \in \mathbb{Q}(X) \mid \phi(\alpha) \in \overline{\mathbb{Z}_p}\}$$

By [25, Proposition 2.2], $\mathbb{Z}_{(p),\alpha}$ is a DVR precisely if α is transcendental over $\mathbb{Q}$, it has rank 2 otherwise. In this last case, if $q \in \mathbb{Q}[X]$ is the minimal polynomial of α , we have $\mathbb{Z}_{(p),\alpha} \subset \mathbb{Q}[X]_{(q)}$.

3.1.2. Residually transcendental extensions. Let W be a DVR of $\mathbb{Q}(X)$ which is residually transcendental over $\mathbb{Z}_{(p)}$ for some $p \in \mathbb{P}$. The next important theorem gives an alternative characterization of these valuations than Theorem 1.6, namely, we show that we can parametrize them by means of pairs of $\overline{\mathbb{Q}_p} \times \mathbb{Q}$ rather than pairs of $\mathbb{Q} \times \mathbb{Q}$.

Theorem 3.4. *Let $p \in \mathbb{P}$ and let W be a residually transcendental extension of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$. Then W is a DVR such that $W = \mathbb{Z}_{p,\alpha,\delta} \cap \mathbb{Q}(X)$ for some $(\alpha, \delta) \in \overline{\mathbb{Q}_p} \times \mathbb{Q}$.*

Proof. We have already said that W is a DVR in Remark 1.3. By Corollary 2.11, W is equal to the restriction to $\mathbb{Q}(X)$ of a (unique) residually transcendental extension $\widetilde{W}$ of $\mathbb{Z}_p$ to $\mathbb{Q}_p(X)$. By Theorem 1.6, $\widetilde{W} = \mathbb{Z}_{p,\alpha,\delta}$ for some $(\alpha, \delta) \in \overline{\mathbb{Q}_p} \times \mathbb{Q}$. Hence, $W = \mathbb{Z}_{p,\alpha,\delta} \cap \mathbb{Q}(X)$. $\square$

The advantage of working over the completion of $\mathbb{Q}$ with respect to the p -adic valuation v_p is that there exists a unique extension of v_p to $\overline{\mathbb{Q}_p}$ since $\mathbb{Q}_p$ is Henselian. Therefore we don't have to worry about fixing an extension of v_p to $\overline{\mathbb{Q}_p}$ in advance (as in Theorem 1.6).

3.1.3. Unitary DVRs of $\mathbb{Q}(X)$. In order to unify the aforementioned two cases, namely, residually algebraic and transcendental extensions of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$, we consider pairs $(\alpha, \delta) \in \mathbb{C}_p \times (\mathbb{Q} \cup \{\infty\})$. In the following we will always assume the following conditions for such a pair:

- if $\delta = \infty$, then α is transcendental over $\mathbb{Q}$.
- if $\delta \in \mathbb{Q}$, then $\alpha \in \overline{\mathbb{Q}_p}$.

The first assumption is due to Theorem 3.1. The second assumption is due to the following fact : in the case of a monomial valuation $v_{p,\alpha,\delta}$ as introduced in section 1.3.1, this valuation can also be defined when $\alpha \in \mathbb{C}_p \setminus \overline{\mathbb{Q}_p}$, but by a density argument we may take $\alpha' \in \overline{\mathbb{Q}_p}$ such that $v_{p,\alpha,\delta} = v_{p,\alpha',\delta}$ (see Remark 3.18 (2) below). Consequently, given a pair $(\alpha, \delta) \in \mathbb{C}_p \times (\mathbb{Q} \cup \{\infty\})$ with $\alpha \in \mathbb{C}_p \setminus \overline{\mathbb{Q}_p}$, automatically $\delta = \infty$. The converse is not true: we can consider pairs (α, ∞) , where α is either in $\overline{\mathbb{Q}_p}$ or in $\mathbb{C}_p \setminus \overline{\mathbb{Q}_p}$.

Remark 3.5. Let $(\alpha, \delta) \in \mathbb{C}_p \times (\mathbb{Q} \cup \{\infty\})$ with the above assumptions. We recall that the value group of $\mathbb{Z}_{(p),\alpha,\delta}$ is of finite index over the value group of $\mathbb{Z}_{(p)}$ (by Remark 1.3 (1) if $\delta \neq \infty$ and by [25, Proposition 2.2] if $\delta = \infty$). In particular, if $\delta \in \mathbb{Q}$, then $\mathbb{Z}_{(p),\alpha,\delta}$ is a DVR. Furthermore, if $\delta = \infty$, by [28, Proposition 2.14, Remark 2.18, Corollary 2.28], $\mathbb{Z}_{(p),\alpha,\delta}$ is a DVR precisely when $\alpha \in \mathbb{C}_p^{\text{br}}$ (we are assuming that α is transcendental over $\mathbb{Q}$). If $\alpha \in \mathbb{C}_p \setminus \mathbb{C}_p^{\text{br}}$, then $\mathbb{Z}_{(p),\alpha,\infty}$ is a one-dimensional non-discrete valuation domain (by the same references).

Given $(\alpha, \delta) \in \mathbb{C}_p \times (\mathbb{Q} \cup \{\infty\})$, we denote by $\mathbb{Z}_{(p),\alpha,\delta}$ the valuation domain $\mathbb{Z}_{p,\alpha,\delta} \cap \mathbb{Q}(X)$. In particular, $\mathbb{Z}_{(p),\alpha} = \mathbb{Z}_{(p),\alpha,\infty}$ by Remark 1.2. The next theorem summarizes the results discussed so far and give a complete description of the unitary DVRs of $\mathbb{Q}(X)$.

Theorem 3.6. *Let W be a DVR of $\mathbb{Q}(X)$ lying above $\mathbb{Z}_{(p)}$, for some $p \in \mathbb{P}$. Then the following statements are satisfied.*

- (1) *$W = \mathbb{Z}_{(p),\alpha,\delta}$ for some $(\alpha, \delta) \in \mathbb{C}_p^{\text{br}} \times (\mathbb{Q} \cup \{\infty\})$. In this case, $X \in W$ if and only if $(\alpha, \delta) \in \mathbb{Q}_{\geq 0}^{\text{br}} \times (\mathbb{Q}_{\geq 0} \cup \{\infty\})$.*

- (2) W is residually transcendental (algebraic, respectively) over $\mathbb{Z}_{(p)}$ if and only if $\delta < \infty$ (α is transcendental over $\mathbb{Q}$ and $\delta = \infty$, respectively).

In particular, we have:

$$\begin{aligned}\mathcal{W}_{p,ra} &= \{\mathbb{Z}_{(p),\alpha} \mid \alpha \in \mathbb{C}_p^{\text{br}} \text{ and } \alpha \text{ is transcendental over } \mathbb{Q}\} \\ \mathcal{W}_{p,rt} &= \{\mathbb{Z}_{(p),\alpha,\delta} \mid (\alpha, \delta) \in \overline{\mathbb{Q}_p} \times \mathbb{Q}\}\end{aligned}$$

Proof. If W is residually transcendental over $\mathbb{Z}_{(p)}$, then there exists $(\alpha, \delta) \in \overline{\mathbb{Q}_p} \times \mathbb{Q}$ such that $W = \mathbb{Z}_{(p),\alpha,\delta}$ (Theorem 3.4). Next if W is residually algebraic over $\mathbb{Z}_{(p)}$, then there exists $\alpha \in \mathbb{C}_p^{\text{br}}$ such that α is transcendental over $\mathbb{Q}$ and $W = \mathbb{Z}_{(p),\alpha}$ (Theorem 3.1).

The second part of item (1) follows by Lemma 1.7. The final claim follows immediately. $\square$

3.2. Polynomial ordering of unitary valuation domains and ultrametric balls in $\mathbb{C}_p$.

For $p \in \mathbb{P}$, we set

$$\mathcal{S}_p = \{\mathbb{Z}_{(p),\alpha,\delta} \mid (\alpha, \delta) \in \mathbb{C}_p \times (\mathbb{Q} \cup \{\infty\})\}.$$

In this subsection, we study under which conditions we have $\mathbb{Z}_{(p),\alpha',\delta'} \cap \mathbb{Q}[X] \subseteq \mathbb{Z}_{(p),\alpha,\delta}$ for $(\alpha, \delta), (\alpha', \delta') \in \mathbb{C}_p \times (\mathbb{Q} \cup \{\infty\})$, and we show that $\mathcal{S}_p$ is a partially ordered set.

Let

$$\mathcal{S} = \left(\bigcup_{p \in \mathbb{P}} \mathcal{S}_p \right) \cup \{\mathbb{Q}[X]_{(q)} \mid q \in \mathbb{Q}[X] \text{ is irreducible}\}.$$

It is noteworthy that if W is a two-dimensional valuation overring of $\mathbb{Z}[X]$ with $W \subset \mathbb{Z}_{(p),0,0}$, then $W \neq \mathbb{Z}_{(p),\alpha,\delta}$ for all $(\alpha, \delta) \in \mathbb{C}_p \times (\mathbb{Q} \cup \{\infty\})$. Therefore $\mathcal{S}$ is not the set of all valuation rings of $\mathbb{Q}(X)$, while $\mathcal{S}$ contains the set of all DVRs of $\mathbb{Q}(X)$ by Theorem 3.6.

Now, in order to show the correspondence between the valuation domains of $\mathcal{S}_p$ and ultrametric balls in $\mathbb{C}_p$, we need to consider ultrametric balls possibly having infinite radius (thus, a singleton). Let $(\alpha, \delta) \in \mathbb{C}_p \times (\mathbb{Q} \cup \{\infty\})$. The ball in $\mathbb{C}_p$ of center α and radius δ is $\bar{B}_p(\alpha, \delta) = \{x \in \mathbb{C}_p \mid v_p(x - \alpha) \geq \delta\}$. Note that $\bar{B}_p(\alpha, \delta) = \{\alpha\}$ precisely when $\delta = \infty$. If $\delta \in \mathbb{Q}$, then $\bar{B}_p(\alpha, \delta)$ has a non-empty intersection with $\overline{\mathbb{Q}_p}$, because $\overline{\mathbb{Q}_p}$ lies dense in $\mathbb{C}_p$, and so we may find an element in $\bar{B}_p(\alpha, \delta) \cap \overline{\mathbb{Q}_p}$ which can be considered as the center of the ball.

We will show that the ordering on $\mathcal{S}_p$ reflects the containment among ultrametric balls in $\mathbb{C}_p$ under the action of the Galois group G_p (Proposition 3.16 and Remarks 3.18). We first need a couple of lemmas. The first lemma uses the notion of pseudo-stationary sequence introduced in [9], see also [26].

Lemma 3.7. *Let $(\alpha, \delta) \in \mathbb{C}_p \times (\mathbb{Q} \cup \{\infty\})$ and suppose that $\bar{B}_p(\alpha, \delta) \subseteq \bigcup_{j=1,\dots,n} B_j$, where the B_j 's are polynomially closed subsets of $\mathbb{C}_p$. Then $\bar{B}_p(\alpha, \delta) \subseteq B_i$ for some $i \in \{1, \dots, n\}$.*

Proof. If $\delta = \infty$, then $\bar{B}_p(\alpha, \delta) = \{\alpha\}$ and the claim clearly is true.

Suppose that $\delta \in \mathbb{Q}$. Let $E = \{s_n\}_{n \in \mathbb{N}} \subset \bar{B}_p(\alpha, \delta)$ be a pseudo-stationary sequence with set of pseudo-limits $\mathcal{L}_E = \bar{B}_p(\alpha, \delta)$. By the pigeonhole principle, a subsequence of E , say $E' = \{s_{n_k}\}_{k \in \mathbb{N}}$, is contained in B_i , for some $i \in \{1, \dots, n\}$; note that E' is still pseudo-stationary with $\mathcal{L}_{E'} = \bar{B}_p(\alpha, \delta)$. By [9, Proposition 4.8], $\bar{B}_p(\alpha, \delta) \subseteq B_i$, as claimed. $\square$

Remark 3.8. More generally, Lemma 3.7 holds for any valuation domain V having infinite residue field; we don't need that the polynomial closure is a topological closure, which in general is not true for valuation domains of arbitrary rank ([30]).

We denote by G_p the Galois group of $\overline{\mathbb{Q}_p}/\mathbb{Q}_p$. We recall that G_p is identified with the group of all continuous $\mathbb{Q}_p$ -automorphism of $\mathbb{C}_p$ (see [5, §3, 1.]) and so G_p acts on $\mathbb{C}_p$. Moreover, for each $\sigma \in G_p$, we have $v_p \circ \sigma = v_p$, because $\mathbb{Q}_p$ is Henselian.

The following remark contains important informations about the set of conjugates of an element $\alpha \in \mathbb{C}_p \setminus \mathbb{Q}_p$ under the action of the group G_p .

Remark 3.9. Let $\alpha \in \mathbb{C}_p$. By [5, Remark 3.2], $G_p(\{\alpha\}) = \{\sigma(\alpha) \mid \sigma \in G_p\}$ is an infinite subset of $\mathbb{C}_p$ precisely when $\alpha \notin \overline{\mathbb{Q}_p}$. In this case, by [5, Theorem 3.5], $G_p(\{\alpha\})$ is totally disconnected, closed and compact (note that this is true also when $G_p(\{\alpha\})$ is a finite set). In particular, by [9, Proposition 2.4], $G_p(\{\alpha\})$ is polynomially closed.

Lemma 3.10. *Let $(\alpha_i, \delta_i) \in \mathbb{C}_p \times (\mathbb{Q} \cup \{\infty\})$, for $i = 1, \dots, n$. Then, $G_p(\bigcup_{i=1}^n \bar{B}_p(\alpha_i, \delta_i))$ is polynomially closed.*

Proof. We remark that

$$(3.11) \quad G_p\left(\bigcup_{i=1}^n \bar{B}_p(\alpha_i, \delta_i)\right) = \bigcup_{i=1}^n G_p(\bar{B}_p(\alpha_i, \delta_i)) = \bigcup_{i=1}^n \bigcup_{\sigma \in G_p} \bar{B}_p(\sigma(\alpha_i), \delta_i).$$

If $\alpha_i \in \overline{\mathbb{Q}_p}$, then $G_p(\bar{B}_p(\alpha_i, \delta_i))$ is equal to a finite union of balls $\bar{B}_p(\sigma(\alpha_i), \delta_i)$, for $\sigma \in G_p$ (which may not be necessarily disjoint); therefore, by [9, Proposition 3.4], $G_p(\bar{B}_p(\alpha_i, \delta_i))$ is polynomially closed. If $\alpha_i \in \mathbb{C}_p \setminus \overline{\mathbb{Q}_p}$ (so δ_i is necessarily equal to ∞ by our convention), then $G_p(\bar{B}_p(\alpha_i, \delta_i)) = G_p(\{\alpha_i\})$ is polynomially closed by Remark 3.9. Hence, $G_p(\bigcup_{i=1}^n \bar{B}_p(\alpha_i, \delta_i))$ is polynomially closed, being a finite union of polynomially closed sets. $\square$

We are now ready to prove the main result of this subsection by which we can show that $\mathcal{S}_p$ is a partially ordered set. We need first the following remark, which gives the connection with rings of integer-valued polynomials.

Remark 3.12. Let $(\alpha, \delta) \in \mathbb{C}_p \times (\mathbb{Q} \cup \{\infty\})$. If $\delta \neq \infty$, given $c \in \overline{\mathbb{Q}_p}$ such that $v_p(c) = \delta$ we have the following equalities:

$$\mathbb{O}_{p,\alpha,\delta} \cap \mathbb{C}_p[X] = \mathbb{O}_p \left[\frac{X - \alpha}{c} \right] = \text{Int}(\bar{B}_p(\alpha, \delta), \mathbb{O}_p).$$

The first equality follows from [26, Remark 3.4] and the second one from [26, Lemma 4.1], since $\mathbb{O}_p$ is a rank one non-discrete valuation domain with infinite residue field. More generally, it is clear that the equality

$$(3.13) \quad \mathbb{O}_{p,\alpha,\delta} \cap \mathbb{C}_p[X] = \text{Int}(\bar{B}_p(\alpha, \delta), \mathbb{O}_p)$$

is true also when $\delta = \infty$. In particular, contracting down the previous equality to $\mathbb{Q}_p[X]$ (and similarly to $\mathbb{Q}[X]$), we have

$$(3.14) \quad \mathbb{Z}_{p,\alpha,\delta} \cap \mathbb{Q}_p[X] = \text{Int}_{\mathbb{Q}_p}(\bar{B}_p(\alpha, \delta), \mathbb{O}_p)$$

Clearly, we have

$$(3.15) \quad \text{Int}(\bar{B}_p(\alpha, \delta), \mathbb{O}_p) = \bigcap_{\alpha' \in \bar{B}_p(\alpha, \delta)} \mathbb{O}_{p,\alpha'} \cap \mathbb{C}_p[X]$$

Proposition 3.16. Let $p \in \mathbb{P}$ and $(\alpha, \delta), (\alpha_i, \delta_i) \in \mathbb{C}_p \times (\mathbb{Q} \cup \{\infty\})$, for $i = 1, \dots, n$. Then the following conditions are equivalent:

- (1) $\mathbb{Z}_{(p),\alpha_i,\delta_i} \cap \mathbb{Q}[X] \subset \mathbb{Z}_{(p),\alpha,\delta}$ for some $i \in \{1, \dots, n\}$.
- (2) $\bigcap_{i=1}^n \mathbb{Z}_{(p),\alpha_i,\delta_i} \cap \mathbb{Q}[X] \subset \mathbb{Z}_{(p),\alpha,\delta}$.
- (3) $\delta \geq \delta_i$ and $v_p(\alpha - \sigma(\alpha_i)) \geq \delta_i$ for some $\sigma \in G_p$ and $i \in \{1, \dots, n\}$.

Proof. (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) By Lemma 2.5, (2) is equivalent to the analogous containment in $\mathbb{Q}_p(X)$:

$$(3.17) \quad \bigcap_{i=1}^n (\mathbb{Z}_{p,\alpha_i,\delta_i} \cap \mathbb{Q}_p[X]) \subset \mathbb{Z}_{p,\alpha,\delta}.$$

By (3.14), the left-hand side of (3.17) is equal to

$$\bigcap_{i=1}^n \text{Int}_{\mathbb{Q}_p}(\bar{B}_p(\alpha_i, \delta_i), \mathbb{O}_p) = \text{Int}_{\mathbb{Q}_p}(B, \mathbb{O}_p),$$

where $B = \bigcup_{i=1}^n \bar{B}_p(\alpha_i, \delta_i)$.

A straightforward adaptation of [27, Lemma 2.20] shows that the integral closure of $\text{Int}_{\mathbb{Q}_p}(B, \mathbb{O}_p)$ in $\overline{\mathbb{Q}_p}(X)$ is equal to $\text{Int}_{\overline{\mathbb{Q}_p}}(G_p(B), \mathbb{O}_p)$. Hence, (3.17) is actually equivalent to

$$\text{Int}_{\overline{\mathbb{Q}_p}}(G_p(B), \mathbb{O}_p) \subset (\overline{\mathbb{Z}_p})_{\alpha,\delta}.$$

By applying Lemma 2.5 again, we can transfer the previous containment over $\mathbb{C}_p$:

$$\text{Int}(G_p(B), \mathbb{O}_p) \subset \mathbb{O}_{p,\alpha,\delta} \cap \mathbb{C}_p[X] = \text{Int}(\bar{B}_p(\alpha, \delta), \mathbb{O}_p)$$

which amounts to say that $\bar{B}_p(\alpha, \delta)$ is contained in the polynomial closure of $G_p(B)$. By Lemma 3.10, $G_p(B)$ is polynomially closed, so $\bar{B}_p(\alpha, \delta) \subseteq G_p(B)$. By Lemma 3.10 again, for each $i \in \{1, \dots, n\}$, $G_p(\bar{B}_p(\alpha_i, \delta_i))$ is polynomially closed, so by Lemma 3.7, $\bar{B}_p(\alpha, \delta) \subseteq G_p(\bar{B}_p(\alpha_i, \delta_i))$ for some $i \in \{1, \dots, n\}$. If $\delta_i = \infty$, then necessarily $\delta = \infty$ and $\alpha = \sigma(\alpha_i)$ for some $\sigma \in G_p$, because otherwise a similar argument to the proof of Lemma 3.7 shows that $G_p(\{\alpha_i\})$ would contain a pseudo-stationary sequence which is not possible by compactness. If $\delta_i < \infty$, again Lemma 3.7 implies that $\bar{B}_p(\alpha, \delta)$ is contained in $\sigma(\bar{B}_p(\alpha_i, \delta_i)) = \bar{B}_p(\sigma(\alpha_i), \delta_i)$ for some $\sigma \in G_p$, which amounts to say that $\delta \geq \delta_i$ and $v_p(\alpha - \sigma(\alpha_i)) \geq \delta_i$, which is precisely condition (3).

(3) $\Rightarrow$ (1) Note that $\mathbb{Z}_{(p), \alpha_i, \delta_i} = \mathbb{Z}_{(p), \sigma(\alpha_i), \delta_i}$ for each $\sigma \in G_p$ by Theorem 1.9. Note also that the inequality $v_p(\alpha - \sigma(\alpha_i)) \geq \delta_i$ by (1.5) is equivalent to $\mathbb{Z}_{(p), \sigma(\alpha_i), \delta_i} = \mathbb{Z}_{(p), \alpha, \delta_i}$. Therefore,

$$\mathbb{Z}_{(p), \alpha_i, \delta_i} \cap \mathbb{Q}[X] = \mathbb{Z}_{(p), \alpha, \delta_i} \cap \mathbb{Q}[X] \subset \mathbb{Z}_{(p), \alpha, \delta}$$

where the last containment follows from the inequality $\delta \geq \delta_i$. $\square$

Remarks 3.18. (1) Let $(\alpha_i, \delta_i) \in \mathbb{C}_p \times (\mathbb{Q} \cup \{\infty\})$, for $i = 1, 2$. By Proposition 3.16, we have the following equivalence:

$$\mathbb{Z}_{(p), \alpha_1, \delta_1} \cap \mathbb{Q}[X] \subset \mathbb{Z}_{(p), \alpha_2, \delta_2} \Leftrightarrow \delta_2 \geq \delta_1 \text{ and } v_p(\alpha_2 - \sigma(\alpha_1)) \geq \delta_1 \text{ for some } \sigma \in G_p.$$

This last condition is equivalent to saying that

$$\bar{B}_p(\alpha_2, \delta_2) \subseteq \sigma(\bar{B}_p(\alpha_1, \delta_1)) = \bar{B}_p(\sigma(\alpha_1), \delta_1).$$

In particular, the residually algebraic DVRs $\mathbb{Z}_{(p), \alpha} = \mathbb{Z}_{(p), \alpha, \infty}$ of $\mathcal{S}_p$ are precisely the maximal elements of $\mathcal{S}_p$ under this ordering (Corollary 2.2 (2)) and each residually transcendental $\mathbb{Z}_{(p), \alpha, \delta} \in \mathcal{S}_p$, with $\delta \in \mathbb{Q}$, is strictly less than a maximal element of $\mathcal{S}_p$, namely, $\mathbb{Z}_{(p), \alpha', \infty}$ for $\alpha' \in \bar{B}_p(\alpha, \delta)$.

(2) Let $(\alpha_i, \delta_i) \in \mathbb{C}_p \times (\mathbb{Q} \cup \{\infty\})$ for $i = 1, 2$ be such that $\mathbb{Z}_{(p), \alpha_1, \delta_1} \cap \mathbb{Q}[X] = \mathbb{Z}_{(p), \alpha_2, \delta_2} \cap \mathbb{Q}[X]$. By Proposition 3.16, we have $\delta_1 = \delta_2$, $v_p(\alpha_2 - \sigma_1(\alpha_1)) \geq \delta_1$, and $v_p(\alpha_1 - \sigma_2(\alpha_2)) \geq \delta_2$ for some $\sigma_1, \sigma_2 \in G_p$. If $\delta_1 = \infty$, then $\alpha_1 = \sigma(\alpha_2)$ for some $\sigma \in G_p$ (that is, α_1, α_2 are conjugate over $\mathbb{Q}_p$), so, in particular, $\mathbb{Z}_{(p), \alpha_1, \delta_1} = \mathbb{Z}_{(p), \alpha_2, \delta_2}$. If instead we have $\delta_i < \infty$ for $i = 1, 2$, then $\mathbb{Z}_{(p), \alpha_1, \delta_1} = \mathbb{Z}_{(p), \sigma(\alpha_1), \delta_1}$ by Theorem 1.9 and $\mathbb{Z}_{(p), \sigma_1(\alpha_1), \delta_1} = \mathbb{Z}_{(p), \alpha_2, \delta_1}$ by (1.5), and hence again we get $\mathbb{Z}_{(p), \alpha_1, \delta_1} = \mathbb{Z}_{(p), \alpha_2, \delta_2}$. Thus, $\mathcal{S}_p$ is a partially ordered set under $\mathbb{Z}_{(p), \alpha_1, \delta_1} \cap \mathbb{Q}[X] \subseteq \mathbb{Z}_{(p), \alpha_2, \delta_2} \cap \mathbb{Q}[X]$ for $\mathbb{Z}_{(p), \alpha_i, \delta_i} \in \mathcal{S}_p$.

Let $\mathcal{B}_p$ be the set of ultrametric balls in $\mathbb{C}_p$ with center in $\mathbb{C}_p^{\text{br}}$ and possibly infinite radius, namely:

$$\mathcal{B}_p = \{\bar{B}_p(\alpha, \delta) \mid (\alpha, \delta) \in \mathbb{C}_p^{\text{br}} \times (\mathbb{Q} \cup \{\infty\})\}$$

with the conventions that

- if $\delta = \infty$, then α is transcendental over $\mathbb{Q}$.
- if $\delta < \infty$, then $\alpha \in \overline{\mathbb{Q}_p}$.

Note that the Galois group $G_p = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ acts on $\mathcal{B}_p$ (Lemma 1.8 and Theorem 1.9), we denote by $\mathcal{B}_p/G_p$ the set of equivalence classes of $\mathcal{B}_p$ under this action. Note that, by Remark 3.9, the cardinality of an equivalence class of a ball $\bar{B}_p(\alpha, \delta)$ under the action of G_p is infinite precisely when $\delta = \infty$ and $\alpha \notin \overline{\mathbb{Q}_p}$.

Theorem 3.19. *The map*

$$\Theta_p : \begin{array}{ccc} \mathcal{B}_p/G_p & \rightarrow & \mathcal{W}_p \\ [\bar{B}_p(\alpha, \delta)] & \mapsto & \mathbb{Z}_{(p), \alpha, \delta} \end{array}$$

is a bijection.

Proof. We show first that Θ_p is well-defined. If $[\bar{B}_p(\alpha, \delta)] = [\bar{B}_p(\alpha', \delta')]$ then there exists $\sigma \in G_p$ such that $\bar{B}_p(\alpha, \delta) = \sigma(\bar{B}_p(\alpha', \delta')) = \bar{B}_p(\sigma(\alpha'), \delta')$, so in particular we get $\delta = \delta'$ and $v_p(\alpha - \sigma(\alpha')) \geq \delta$. Hence, by Proposition 3.16, we have $\mathbb{Z}_{(p), \alpha, \delta} \cap \mathbb{Q}[X] = \mathbb{Z}_{(p), \alpha', \delta'} \cap \mathbb{Q}[X]$ and thus by Remark 3.18 (2) we get $\mathbb{Z}_{(p), \alpha, \delta} = \mathbb{Z}_{(p), \alpha', \delta'}$.

For the injectivity of Θ_p , if the equality $\mathbb{Z}_{(p), \alpha, \delta} = \mathbb{Z}_{(p), \alpha', \delta'}$ holds, then in particular $\mathbb{Z}_{(p), \alpha, \delta} \cap \mathbb{Q}[X] = \mathbb{Z}_{(p), \alpha', \delta'} \cap \mathbb{Q}[X]$ and the same arguments as above show that $[\bar{B}_p(\alpha, \delta)] = [\bar{B}_p(\alpha', \delta')]$.

The surjectivity of Θ_p follows from Theorem 3.6. $\square$

4. POLYNOMIAL KRULL DOMAINS

We are interested in characterizing polynomial Krull Domains, i.e., Krull domains R such that $\mathbb{Z}[X] \subseteq R \subseteq \mathbb{Q}[X]$. Note that since $\mathbb{Z}[X]$ is a 2-dimensional Noetherian domain, by [18, Theorem 9] such a Krull domain R is an integrally closed Noetherian domain of Krull dimension at most 2. Conversely, it is well-known that every integrally closed Noetherian domain is a Krull domain (for example, see [32, Chapter VI, §13, (b), p. 82]). Hence, we are describing the family of integrally closed Noetherian domains between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$. Given a prime ideal Q of an integral domain R as above, we say that Q is *unitary* if $Q \cap \mathbb{Z} = p\mathbb{Z}$ for some $p \in \mathbb{P}$ and that Q is *non-unitary* if $Q \cap \mathbb{Z} = (0)$.

4.1. Representations of polynomial Krull domains. We recall that we have the following representation of the PID $\mathbb{Q}[X]$ as an intersection of DVRs:

$$\mathbb{Q}[X] = \bigcap_{q \in \mathcal{P}^{\text{irr}}} \mathbb{Q}[X]_{(q)}$$

where $\mathcal{P}^{\text{irr}}$ is the set of irreducible polynomials of $\mathbb{Q}[X]$ and $\mathbb{Q}[X]_{(q)}$ is the localization of $\mathbb{Q}[X]$ at the prime ideal generated by $q(X)$.

In general, let R be a polynomial Krull domain, $\Lambda = \{p \in \mathbb{P} \mid pR \subset R\}$, $X^1(R)$ be the set of height one prime ideals of R , and $X_p^1(R) = \{Q \in$

$X^1(R) \mid p \in Q\}$ for each $p \in \Lambda$. Then $X_p^1(R)$ is finite, R_Q is a DVR for all $Q \in X_p^1(R)$, and

$$R = \bigcap_{p \in \Lambda} \left(\bigcap_{Q \in X_p^1(R)} R_Q \right) \cap \mathbb{Q}[X].$$

Hence, for each $p \in \Lambda$, there exists a finite set $\{W_{p,1}, \dots, W_{p,m_p}\}$ of DVRs with quotient field $\mathbb{Q}(X)$, so that $W_{p,j} \cap \mathbb{Q} = \mathbb{Z}_{(p)}$ for $j = 1, \dots, m_p$, and

$$(4.1) \quad R = \bigcap_{p \in \Lambda} \left(\bigcap_{j=1}^{m_p} W_{p,j} \right) \cap \mathbb{Q}[X] = \bigcap_{p \in \Lambda} \left(\bigcap_{j=1}^{m_p} (W_{p,j} \cap \mathbb{Q}[X]) \right).$$

For each $p \in \Lambda$, let

$$(4.2) \quad R_p = \bigcap_{j=1}^{m_p} (W_{p,j} \cap \mathbb{Q}[X])$$

It is clear that R_p is a Krull domain between $\mathbb{Z}_{(p)}[X]$ and $\mathbb{Q}[X]$ and $R = \bigcap_{p \in \Lambda} R_p$.

Lemma 4.3. *Let $p \in \mathbb{P}$ and R be a ring between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ such that $R = \bigcap_{Q \in X^1(R)} R_Q$ and R_Q is a DVR for all $Q \in X^1(R)$, $X_p^1(R) = \{Q \in X^1(R) \mid p \in Q\}$ and $R_p = \left(\bigcap_{Q \in X_p^1(R)} R_Q \right) \cap \mathbb{Q}[X]$. Then the following statements hold true.*

- (1) $R = \bigcap_{p \in \mathbb{P}} R_p$.
- (2) If $R_{(p)} = (\mathbb{Z} \setminus p\mathbb{Z})^{-1}R$, then $R_{(p)} = R_p$.
- (3) $R_{(p)} \cap \mathbb{Q} = \mathbb{Z}_{(p)}$.

Proof. (1) If Q is a prime ideal of R , either $Q \cap \mathbb{Z} = (0)$ or $Q \cap \mathbb{Z} = p\mathbb{Z}$ for some $p \in \mathbb{P}$. Moreover, if $Q \cap \mathbb{Z} = (0)$, then $\mathbb{Q}[X] \subseteq R_Q$. Hence, $R = R \cap \mathbb{Q}[X] = \bigcap_{p \in \mathbb{P}} \left(\left(\bigcap_{Q \in X_p^1(R)} R_Q \right) \cap \mathbb{Q}[X] \right) = \bigcap_{p \in \mathbb{P}} R_p$.

(2) ($\subseteq$) Let $\frac{f}{n} \in R_{(p)}$, where $f \in R$ and $n \in \mathbb{Z}$ is not divisible by p . Then clearly $\frac{f}{n} \in R_Q$, for every $Q \in X_p^1(R)$ since the value of each of the corresponding valuation of n is zero and $R \subseteq R_p$. Thus, $R_{(p)} \subseteq R_p$. ($\supseteq$) Conversely, let $h \in R_p$. Then we may write $h = \frac{g}{mp^k}$, for some $g \in \mathbb{Z}[X]$, $m \in \mathbb{Z}$ not divisible by p and $k \geq 0$. Then $mh = \frac{g}{p^k}$ is clearly in R (because for each $q \in \mathbb{P}$, $q \neq p$, p^k is a unit of R_Q for all $Q \in X_q^1(R)$). Thus, we also have $R_{(p)} \supseteq R_p$.

(3) The result follows from the following two observations: (i) $R_Q \cap \mathbb{Q} = \mathbb{Z}_{(p)}$ for all $Q \in X_p^1(R)$ and (ii) $R_{(p)} = R_p$ by (2). $\square$

Let R be as in Lemma 4.3 and $R'_p = \bigcap_{Q \in X_p^1(R)} R_Q$ for all $p \in \mathbb{P}$. Then R is a Krull domain if and only if R_p is a Krull domain for all $p \in \mathbb{P}$ and, for any $0 \neq f \in \mathbb{Z}[X]$, $fR'_p = R'_p$ for all but a finitely many p 's in $\mathbb{P}$. Hence, in the remaining part of this paper, we first study when R_p is a Krull domain and we then study the Krull domain R by investigating the property that if $f \in \mathbb{Z}[X]$ is nonzero, then $fR'_p = R'_p$ for all but a finitely many p 's in $\mathbb{P}$. The first case will be called “Local case” and the second one “Global case”.

4.2. Local case. Let $p \in \mathbb{P}$ be fixed. In this section we consider a Krull domain R between $\mathbb{Z}_{(p)}[X]$ and $\mathbb{Q}[X]$; we show that R can be represented as a ring of integer-valued polynomials over a finite union of ultrametric balls in $\mathbb{C}_p$. Clearly, for every such R we have

$$(4.4) \quad R = \bigcap_{j=1}^m W_j \cap \mathbb{Q}[X]$$

where $\{W_i \mid i = 1, \dots, m\}$ is a finite family (possibly empty) of DVRs of $\mathbb{Q}(X)$ lying above $\mathbb{Z}_{(p)}$.

4.2.1. Irredundant representation. We first investigate when a representation of a Krull domain between $\mathbb{Z}_{(p)}[X]$ and $\mathbb{Q}[X]$ as in (4.4) is irredundant. Let D be an integral domain with quotient field K and suppose that $D = \bigcap_{i \in I} V_i$, where $\{V_i\}_{i \in I}$ is a family of valuation domains of K (by Krull's Theorem this is equivalent to say that D is integrally closed in K). As in [15], we say that an element $V = V_{i_0}$ of this family is superfluous if $D = \bigcap_{i \in I, i \neq i_0} V_i$ and that the representation $D = \bigcap_{i \in I} V_i$ is irredundant if no element V_i of the family $\{V_i\}_{i \in I}$ is superfluous.

The following result is classical, see for example [23, Remark on page 296] or [14, Corollary 43.9].

Theorem 4.5. *Let D be a Krull domain. Then $\{D_P \mid P \in X^1(D)\}$ is the unique defining family for D , that is, if $\{V_\alpha\}$ is a defining family for D , then $\{D_P \mid P \in X^1(D)\} \subseteq \{V_\alpha\}$.*

Let D be a Krull domain. Theorem 4.5 also says that if Λ is a subset of $X^1(D)$, then $D \subsetneq \bigcap_{P \in \Lambda} D_P$ if and only if $\Lambda \subsetneq X^1(D)$. In particular, for every $P \in X^1(D)$, D_P is not superfluous for the defining family $\{D_P \mid P \in X^1(D)\}$, which therefore realizes the unique irredundant representation of D .

Corollary 4.6. *Let D be a Krull domain, $\{V_\alpha\}$ be a defining family for D and $V \in \{V_\alpha\}$ with maximal ideal M .*

- (1) *V is superfluous in $D = \bigcap_{\alpha} V_\alpha$ if and only if $ht(M \cap D) \geq 2$.*

- (2) The intersection $D = \bigcap_{\alpha} V_{\alpha}$ is irredundant if and only if $\text{ht}(M_{\alpha} \cap D) = 1$ for all α .

Proof. (1) Suppose that V is not superfluous in $D = \bigcap_{\alpha} V_{\alpha}$ and let $R = \bigcap_{V_{\alpha} \neq V} V_{\alpha}$. Then $D = R \cap V$ and $R \not\subseteq V$. Hence $V = D_{M \cap D}$ [19, Lemma 1.3], and since V is a DVR, $\text{ht}(M \cap D) = \text{ht}((M \cap D)D_{M \cap D}) = 1$. Conversely, assume that $\text{ht}(M \cap D) < 2$. Then $\text{ht}(M \cap D) = 1$, and hence $D_{M \cap D}$ is a DVR. Note that $D_{M \cap D} \subseteq V$, so $D_{M \cap D} = V$. Thus V is not superfluous in $D = \bigcap_{\alpha} V_{\alpha}$.

(2) This follows directly from (1). $\square$

Corollary 4.7. *Let R be as in (4.4) and let $W = W_j$ for some $j \in \{1, \dots, m\}$. If W' is a DVR extending $\mathbb{Z}_{(p)}$ such that $w < w'$, then $R \subset W'$ and the center of W' on R has height two.*

Moreover, if W'' is another DVR extending $\mathbb{Z}_{(p)}$ such that $w < w' \leq w''$, then $M \cap R \subset M' \cap R = M'' \cap R$.

Proof. By assumption that $w < w'$, we have $R \subseteq W \cap \mathbb{Q}[X] \subseteq W' \cap \mathbb{Q}[X]$. Hence W' is superfluous in $R \cap W'$, and since $W \neq W'$, we have $\text{ht}(M' \cap R) \geq 2$ by Corollary 4.6. Note that $\dim(R) \leq 2$. Thus $\text{ht}(M' \cap R) = 2$.

For the last statement, we have $R \subset W' \cap \mathbb{Q}[X] \subseteq W'' \cap \mathbb{Q}[X]$, so $M' \cap R \subseteq M'' \cap R$. Since $\dim(R) \leq 2$, equality holds in the previous containment. $\square$

Corollary 4.8. *Let R be as in (4.4). Then, for every $q \in \mathbb{Q}[X]$ irreducible, $\mathbb{Q}[X]_{(q)}$ is not superfluous in that representation of R .*

Proof. Let Q be the center of $\mathbb{Q}[X]_{(q)}$ on R . Then $Q \cap \mathbb{Z} = (0)$, and hence if $S = \mathbb{Z} \setminus \{0\}$, then $R_S = \bigcap_{i=1}^n (W_i)_S \cap \mathbb{Q}[X] = \mathbb{Q}[X]$ [14, Proposition 43.5] and $QR_S \subsetneq R_S$. Thus $\text{ht}Q = \text{ht}(QR_S) = 1$, so, by Corollary 4.6, 2., $\mathbb{Q}[X]_{(q)}$ is not superfluous. $\square$

Remark 4.9. (1) By Proposition 3.16, geometrically the representation of R in (4.4) is irredundant if and only if the balls $\bar{B}_p(\alpha_i, \delta_i)$, $(\alpha_i, \delta_i) \in \mathbb{O}_p^{\text{br}} \times (\mathbb{Q}_{\geq 0} \cup \{\infty\})$, for $i = 1, \dots, n$, are disjoint under the action of the Galois group G_p in the following sense: for each pair of balls $\bar{B}_p(\alpha_{i_1}, \delta_{i_1}), \bar{B}_p(\alpha_{i_2}, \delta_{i_2})$, there is no $\sigma \in G_p$ such that $\bar{B}_p(\alpha_{i_1}, \delta_{i_1})$ is contained in $\sigma(\bar{B}_p(\alpha_{i_2}, \delta_{i_2}))$ (see Remarks 3.18). Equivalently, the balls $\bar{B}_p(\sigma(\alpha_i), \delta_i)$, for $i = 1, \dots, n$ and $\sigma \in G_p$, are disjoint. We stress that the number of such balls is finite if and only if there is no $j \in \{1, \dots, n\}$ such that $\alpha_j \in \mathbb{O}_p^{\text{br}} \setminus \overline{\mathbb{Z}_p}$ (see Remark 3.9).

(2) By Corollary 4.6 and Corollary 4.8, the representation of R in (4.4) is irredundant if and only if $M_j \cap R$ has height one for $j = 1, \dots, m$, where M_j is the maximal ideal of W_j .

Now we study some basic ring-theoretic properties of $R = \bigcap_{j=1}^m W_j \cap \mathbb{Q}[X]$, which are very useful for the study of polynomial Krull domains. In particular, we characterize when the prime unitary ideals of R are maximal. If W is a DVR of $\mathbb{Q}(X)$ extending $\mathbb{Z}_{(p)}$ for some $p \in \mathbb{P}$, we denote by $e(W | \mathbb{Z}_{(p)})$ the ramification index of the extension $W \supset \mathbb{Z}_{(p)}$. If $\alpha \in \mathbb{C}_p^{\text{br}}$, we denote by $e(\mathbb{Q}_p(\alpha) | \mathbb{Q}_p)$ the ramification index of the extension $\mathbb{Q}_p(\alpha) \supseteq \mathbb{Q}_p$ (we stress that this extension is algebraic precisely when $\alpha \in \overline{\mathbb{Q}_p}$).

Theorem 4.10. *Let $p \in \mathbb{P}$. Suppose that $W_j = \mathbb{Z}_{(p), \alpha_j, \delta_j}$, for some $(\alpha_j, \delta_j) \in \mathbb{C}_p^{\text{br}} \times (\mathbb{Q} \cup \{\infty\})$, is a DVR of $\mathbb{Q}(X)$ extending $\mathbb{Z}_{(p)}$ and $X \in W_j$ for $j = 1, \dots, m$. Now, let $M_j = M_{(p), \alpha_j, \delta_j}$ for each $j = 1, \dots, m$, and*

$$(4.11) \quad R = \bigcap_{j=1}^m W_j \cap \mathbb{Q}[X].$$

If the representation of R in (4.11) is irredundant (namely, $\text{ht}(M_j \cap R) = 1$ for $j = 1, \dots, m$), then the following statements hold:

- (1) *R is a Krull domain such that $\mathbb{Z}_{(p)}[X] \subseteq R \subseteq \mathbb{Q}[X]$.*
- (2) *$X^1(R) = \{M_j \cap R \mid j = 1, \dots, m\} \cup \{f\mathbb{Q}[X] \cap R \mid f \in \mathbb{Q}[X] \text{ is irreducible}\}$.*
- (3) *$M_j \cap R$ is a maximal ideal of R if and only if W_j is residually algebraic over $\mathbb{Z}_{(p)}$.*
- (4) *Suppose that (α_j, δ_j) is a minimal pair if $\delta_j \in \mathbb{Q}$. Then we have*

$$e(W_j | \mathbb{Z}_{(p)}) = \begin{cases} e(\mathbb{Q}_p(\alpha_j) | \mathbb{Q}_p), & \text{if } \delta_j = \infty \\ e(\mathbb{Q}_p(\alpha_j) | \mathbb{Q}_p) e(\gamma_j | \mathbb{Q}_p(\alpha_j)), & \text{if } \delta_j < \infty \end{cases}$$

where $\gamma_j = w_j(f_{\alpha_j})$, f_{α_j} is the minimal polynomial of α_j over $\mathbb{Q}_p$ and $e(\gamma_j | \mathbb{Q}_p(\alpha_j))$ is the smallest natural number n such that $n\gamma_j$ is in the value group of $\mathbb{Q}_p(\alpha_j)$.

- (5) *If $e_j = e(W_j | \mathbb{Z}_{(p)})$ for $j = 1, \dots, m$ and $d = \gcd(e_1, \dots, e_m)$, then $\text{Cl}(R) = \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}^{m-1}$.*
- (6) *R is Dedekind if and only if $\delta_j = \infty$ and α_i is transcendental over $\mathbb{Q}$ for each $j = 1, \dots, m$.*

Proof. (1) Clear, since R is a finite intersection of Krull domains with the same quotient field ([7, Chapter VII, §1, 3. Exemples, 3]).

(2) The containment $(\supseteq)$ follows by [10, Corollary 2.3(3)] and the assumption that $M_j \cap R \in X^1(R)$ for all $j \in \{1, \dots, m\}$. The other containment follows easily by Theorem 4.5.

(3) If W_j is residually transcendental over $\mathbb{Z}_{(p)}$, that is, $\delta_j \in \mathbb{Q}$ (Theorem 3.6), then the claim follows by Corollary 4.7. Suppose instead that W_j is residually algebraic over $\mathbb{Z}_{(p)}$, that is, $\delta_j = \infty$ (Theorem 3.6). We have the following containments: $\mathbb{Z}/p\mathbb{Z} \subseteq R/(M_j \cap R) \subseteq W_j/M_j$; since W_j/M_j is algebraic over $\mathbb{Z}/p\mathbb{Z}$, it follows that $M_j \cap R$ is a maximal ideal of R .

(4) If $\delta_j < \infty$, see [2, Theorem 2.1, c)], while if $\delta_j = \infty$, see [25, Proposition 2.2].

(5) Let G be a free abelian group of rank m on the generators $g_1, \dots, g_m$ and H be the subgroup of G generated by $r = \sum_{i=1}^m e_i g_i$. Then $\text{Cl}(R) = G/H$ [10, Theorem 2.4], and since $d = \gcd(e_1, \dots, e_m)$, $\text{Cl}(R) = \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}^{m-1}$ [10, Remark 1.6].

(6) A Krull domain is Dedekind if and only if it has (Krull) dimension one ([32, Theorem 28, Chapt. VI, §13]). The proof then follows from items (2), (3) and Theorem 3.6. $\square$

Definition 4.12. An infinite family $\{W_i\}_{i \in I}$ of DVRs of $\mathbb{Q}(X)$ will be said to be *polynomially finite* if there exists a finite set $\{V_1, \dots, V_n\}$ of DVRs of $\mathbb{Q}(X)$ such that $\bigcap_{i \in I} W_i \cap \mathbb{Q}[X] = \bigcap_{j=1}^n V_j \cap \mathbb{Q}[X]$.

For example, putting together (3.13) and (3.15), we have

$$\bigcap_{\alpha' \in \bar{B}_p(\alpha, \delta)} \mathbb{O}_{p, \alpha', \infty} \cap \mathbb{C}_p[X] = \mathbb{O}_{p, \alpha, \delta} \cap \mathbb{C}_p[X].$$

Note that an element of the family $\{\mathbb{O}_{p, \alpha', \infty} \mid \alpha' \in \bar{B}_p(\alpha, \delta)\}$ is a DVR if and only if α' is transcendental over $\mathbb{Q}$ ([25, Proposition 2.2]), so in particular, not every of the valuation overring $\mathbb{O}_{p, \alpha_p, \infty}$ above is a DVR.

The next theorem establishes when an infinite family $\{W_i\}_{i \in I}$ of DVRs of $\mathbb{Q}(X)$ extending $\mathbb{Z}_{(p)}$ is polynomially finite.

Theorem 4.13. *Let $p \in \mathbb{P}$. Suppose that $\mathbb{Z}_{(p), \alpha_i, \delta_i}$, for some $(\alpha_i, \delta_i) \in \mathbb{C}_p^{br} \times (\mathbb{Q} \cup \{\infty\})$, is an infinite family of DVRs of $\mathbb{Q}(X)$ extending $\mathbb{Z}_{(p)}$, for $i \in I$, and let $R = \bigcap_{i \in I} \mathbb{Z}_{(p), \alpha_i, \delta_i} \cap \mathbb{Q}[X]$. Then the following statements are equivalent:*

- (1) *R is a Krull domain.*
- (2) *The set $\{\mathbb{Z}_{(p), \alpha_i, \delta_i} \mid i \in I\}$ is polynomially finite.*
- (3) *$R = \bigcap_{j=1}^n \mathbb{Z}_{(p), \alpha_j, \delta_j} \cap \mathbb{Q}[X]$ for some $\{(\alpha_j, \delta_j) \mid j = 1, \dots, n\} \subseteq \mathbb{C}_p^{br} \times (\mathbb{Q} \cup \{\infty\})$.*
- (4) *There exists a finite subset $\{(\alpha_j, \delta_j) \mid j = 1, \dots, n\}$ of $\mathbb{C}_p^{br} \times (\mathbb{Q} \cup \{\infty\})$, so that $R \subseteq \bigcap_{j=1}^n \mathbb{Z}_{(p), \alpha_j, \delta_j}$ and for each $i \in I$, there exists $j \in \{1, \dots, n\}$ such that $\delta_i \geq \delta_j$ and $v(\alpha_i - \sigma(\alpha_j)) \geq \delta_j$, for some $\sigma \in G_p$.*

Proof. (1) $\Rightarrow$ (2) Let $\mathcal{A} = X_p^1(R)$. Then $\mathcal{A}$ is finite, $R = \bigcap_{Q \in \mathcal{A}} R_Q \cap \mathbb{Q}[X]$, R_Q is a DVR of $\mathbb{Q}(X)$ extending $\mathbb{Z}_{(p)}$. Thus $\{\mathbb{Z}_{(p), \alpha_i, \delta_i} \mid i \in I\}$ is polynomially finite.

(2) $\Rightarrow$ (3) This follows from the fact that each DVR of $\mathbb{Q}(X)$ extending $\mathbb{Z}_{(p)}$ is of the form $\mathbb{Z}_{(p),\alpha,\delta}$ for some $(\alpha, \delta) \in \mathbb{C}_p^{\text{br}} \times (\mathbb{Q} \cup \{\infty\})$ by Theorem 3.6.

(3) $\Rightarrow$ (1) Clear, since R is a finite intersection of Krull domains with the same quotient field ([7, Chapter VII, §1, 3. Exemples, 3]).

(3) $\Rightarrow$ (4) Each $\mathbb{Z}_{(p),\alpha_i,\delta_i}$ is an overring of $\bigcap_{j=1}^n \mathbb{Z}_{(p),\alpha_j,\delta_j} \cap \mathbb{Q}[X]$, so by Proposition 3.16, $\delta_i \geq \delta_j$ and $v(\alpha_i - \sigma(\alpha_j)) \geq \delta_j$, for some $j \in \{1, \dots, m\}$ and $\sigma \in G_p$.

(4) $\Rightarrow$ (3) Suppose that for each $i \in I$, there exists $j \in \{1, \dots, n\}$ such that $\delta_i \geq \delta_j$ and $v(\alpha_i - \sigma(\alpha_j)) \geq \delta_j$, for some $\sigma \in G_p$. Then by Proposition 3.16, $\mathbb{Z}_{(p),\alpha_i,\delta_i}$ is an overring of $\bigcap_{j=1}^n \mathbb{Z}_{(p),\alpha_j,\delta_j} \cap \mathbb{Q}[X]$, so the containment ($\supseteq$)

holds and the equality follows because $R \subseteq \bigcap_{j=1}^n \mathbb{Z}_{(p),\alpha_j,\delta_j}$. $\square$

Remark 4.14. Geometrically, condition 4. of Theorem 4.13 means that the union of the balls $\bar{B}_p(\alpha_i, \delta_i)$ in $\overline{\mathbb{Q}_p}$, for $i \in I$ (each of them corresponds to a valuation domain $\mathbb{Z}_{(p),\alpha_i,\delta_i}$), is contained in finitely many of them up to the action of the Galois group G_p , namely:

$$\bigcup_{i \in I} \bar{B}_p(\alpha_i, \delta_i) \subseteq \bigcup_{j=1, \dots, m} \bigcup_{\sigma \in G_p} \bar{B}_p(\sigma(\alpha_j), \delta_j).$$

We can now characterize Krull Domains R between $\mathbb{Z}_{(p)}[X]$ and $\mathbb{Q}[X]$. By Theorem 3.6, we may restrict to consider in the defining family of DVRs of R only those DVRs extending $\mathbb{Z}_{(p)}$ defined by pairs $(\alpha, \delta) \in \mathbb{O}_p^{\text{br}} \times (\mathbb{Q}_{\geq 0} \cup \{\infty\})$. We recall that $\mathbb{Z}_{(p)}[X]$ is a Krull domain (for example see [13, Proposition 1.6]). If $\mathbb{Z}_{(p),0,0}$ is the valuation domain of the trivial Gaussian extension, then $\mathbb{Z}_{(p)}[X] = \mathbb{Z}_{(p),0,0} \cap \mathbb{Q}[X] = \text{Int}_{\mathbb{Q}}(\mathbb{O}_p)$.

Theorem 4.15. *Let $p \in \mathbb{P}$ and R be a Krull domain between $\mathbb{Z}_{(p)}[X]$ and $\mathbb{Q}[X]$. Then there exist finitely many pairs $(\alpha_i, \delta_i) \in \mathbb{O}_p^{\text{br}} \times (\mathbb{Q}_{\geq 0} \cup \{\infty\})$, $i = 1, \dots, n$, with α_i transcendental over $\mathbb{Q}$ if $\delta_i = \infty$, such that*

$$(4.16) \quad R = \bigcap_{i=1}^n \mathbb{Z}_{(p),\alpha_i,\delta_i} \cap \mathbb{Q}[X] = \text{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p),$$

where $E_p = \bigcup_{i=1}^n \bar{B}_p(\alpha_i, \delta_i)$. Moreover, if the representation in (4.16) is irredundant (see Remark 4.9), then R is Dedekind if and only if either $n = 0$ (and then $R = \mathbb{Q}[X]$) or $n \geq 1$ and $\delta_i = \infty$ for each $i = 1, \dots, n$; thus, in either case, R is Dedekind if and only if E_p contains finitely many elements.

Proof. We remarked at the beginning of this section in (4.4) that $R = \bigcap_{i=1}^n W_i \cap \mathbb{Q}[X]$, where W_i is a DVR of $\mathbb{Q}(X)$ containing $\mathbb{Z}_{(p)}[X]$, for each

$i = 1, \dots, n$. By Theorem 3.6, there exist $(\alpha_i, \delta_i) \in \mathbb{O}_p^{\text{br}} \times (\mathbb{Q}_{\geq 0} \cup \{\infty\})$ such that $W_i = \mathbb{Z}_{(p), \alpha_i, \delta_i}$, for each $i = 1, \dots, n$. Note that, if $\delta_i = \infty$, then necessarily α must be transcendental over $\mathbb{Q}$ (§3.1.3). The representation of R as a ring of integer-valued polynomials follows by (3.14).

Clearly, $R \cap \mathbb{Q}$ is equal either to $\mathbb{Q}$ or to $\mathbb{Z}_{(p)}$ according to whether the family of DVRs W_i is empty or not. In the former case, $R = \mathbb{Q}[X]$ which is a PID (thus, Dedekind and a UFD). In the latter case, by [26, Theorem 3.14], the Krull domain R is Dedekind if and only if there is no W_i which is residually transcendental over $\mathbb{Z}_{(p)}$. This corresponds to saying that $\delta_i = \infty$ for each $i = 1, \dots, n$, by Theorem 3.6. $\square$

4.2.2. Non-unitary prime ideals. Let R be a polynomial Krull domain between $\mathbb{Z}_{(p)}[X]$ and $\mathbb{Q}[X]$ as in (4.4). Let $q \in \mathbb{Q}[X]$ be irreducible and let $Q = q(X)\mathbb{Q}[X]_{(q)} \cap R$. By Theorem 4.10 (2), Q is an height 1 prime ideal of R and $R_Q = \mathbb{Q}[X]_{(q)}$. In this section we characterize when Q is a maximal ideal of R .

In general, we remark that if W is a valuation domain of the field of rational functions $K(X)$ over a field K and W has dimension at least 2, then $W \cap K = V$ is a valuation domain of K which is not K itself. In fact, $W \cap K = K \Leftrightarrow K \subset W$ and in this case W would be a DVR ([11, p. 3]).

Proposition 4.17. *Let W be a valuation domain of $K(X)$ such that $W \subset K[X]_{(q)}$ for some irreducible $q \in K[X]$. Let $V = W \cap K$. Then, there exist a root $\alpha \in \overline{K}$ of $q(X)$ and an extension U of V to $K(\alpha)$ such that $W = U_\alpha = \{\phi \in K(X) \mid \phi(\alpha) \in U\}$.*

Proof. Let $\pi : K[X]_{(q)} \rightarrow K[X]/(q) \cong K(\alpha)$ be the canonical residue map, where α is the residue class of X and can be identified to a root of q . Note that for $\phi \in K[X]_{(q)}$, $\pi(\phi) = \phi(\alpha)$ because π is an homomorphism.

It is well-known that every valuation domain W strictly contained in $K[X]_{(q)}$ is of the form $\pi^{-1}(U)$, where U is a valuation domain of $K(\alpha)$; moreover, $U = \pi(W) = W/Q$, where Q is the maximal ideal of $K[X]_{(q)}$. Hence,

$$(4.18) \quad U = W/Q = \{\phi(\alpha) \mid \phi \in W\} = \{\phi(\alpha) \in K(\alpha) \mid \phi(\alpha) \in U\}.$$

(Actually, every element of $K(\alpha)$ is of the form $g(\alpha)$ for some $g \in K[X]$). Moreover, the maximal ideal of U is equal to M_W/Q , where M_W is the maximal ideal of W . In particular, $U \cap K = V$. In fact, if $\beta \in V = W \cap K$, then $\pi(\beta) = \beta$ because $\pi|_K$ is the identity. Hence, $\beta = \pi(\beta) \in U = W/Q$. Similary, one can show that $M_V \subseteq M_U = M_W/Q$, the maximal ideal of U . Therefore, $V = U \cap K$.

Finally, by (4.18) and the above considerations,

$$W = \pi^{-1}(U) = \{\phi \in K(X) \mid \phi(\alpha) \in U\} = U_\alpha.$$

$\square$

We remark that if we consider the completion of $K(\alpha)$ with respect to the valuation domain U , we may consider α as an element of $\widehat{K}$, algebraic over K , so that we have the equality:

$$W = \{\phi \in K(X) \mid \phi(\alpha) \in \widehat{U}\}$$

where $\widehat{U}$ is the completion of U .

Theorem 4.19. *Let $p \in \mathbb{P}$ and let $R = \bigcap_{j=1}^m \mathbb{Z}_{(p), \alpha_j, \delta_j} \cap \mathbb{Q}[X]$ be a Krull domain, where $(\alpha_j, \delta_j) \in \mathbb{C}_p \times (\mathbb{Q} \cup \{\infty\})$ for $j = 1, \dots, m$. Let $Q = q(X)\mathbb{Q}[X] \cap R$ be a non-unitary prime ideal of R , for some irreducible $q \in \mathbb{Q}[X]$. Then Q is a maximal ideal of R if and only if q has no root in $\bigcup_{j=1}^m \bar{B}_p(\alpha_j, \delta_j)$.*

Proof. Note that when R is Dedekind (that is, $\delta_j = \infty$ for $j = 1, \dots, m$ by Theorem 4.10, (7)), then Q is automatically maximal. In this case each ball $\bar{B}_p(\alpha_j, \delta_j)$ is the singleton $\{\alpha_j\}$ which is necessarily transcendental over $\mathbb{Q}$ because otherwise $\mathbb{Z}_{(p), \alpha_j, \infty}$ would not be a DVR.

In general, suppose that $q(\alpha') = 0$ for some $\alpha' \in \bigcup_{j=1}^m \bar{B}_p(\alpha_j, \delta_j)$. Then, we have these containments:

$$R \subset \mathbb{Z}_{(p), \alpha'} = \mathbb{Z}_{(p), \alpha', \infty} \subset \mathbb{Q}[X]_{(q)}$$

where the first containment follows by Proposition 3.16 and the second one from the fact that $\mathbb{Q}[X]_{(q)} = \{\phi \in K(X) \mid \phi(\alpha') \in \overline{\mathbb{Q}_p}\}$. In particular, $Q \subset M_{(p), \alpha'} \cap R$, so Q is not a maximal ideal.

Conversely, suppose that $Q \subset M \subset R$ for some height 2 maximal ideal M of R , which necessarily contains p . By [14, Corollary 19.7], there exists a 2-dimensional valuation overring W of R with non-zero prime ideals $Q' \subset M'$ such that $Q' \cap R = Q$ and $M' \cap R = M$. Note that $W \cap \mathbb{Q} = \mathbb{Z}_{(p)}$ and $W_{Q'} = R_Q = \mathbb{Q}[X]_{(q)}$. By Proposition 4.17, there exist an extension U of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}[X]/(q)$ and a root α' of $q(X)$ such that $W = \{\phi \in \mathbb{Q}(X) \mid \phi(\alpha') \in U\}$. By the remark after that Proposition, we may assume that $\alpha' \in \overline{\mathbb{Q}_p}$, necessarily algebraic over $\mathbb{Q}$, so that $W = \mathbb{Z}_{(p), \alpha'} = \mathbb{Z}_{(p), \alpha', \infty}$. Since $R \subset \mathbb{Z}_{(p), \alpha'}$, it follows by Proposition 3.16 that α' (or possibly some of its conjugate) is in $\bigcup_{j=1}^m \bar{B}_p(\alpha_j, \delta_j)$, as claimed. $\square$

Corollary 4.20. *Let R and Q be as in Theorem 4.19 and suppose that Q is not a maximal ideal. Then the maximal ideals of R containing Q are unitary and they are finitely many.*

Proof. If $Q \subset M \subset R$ where M is a maximal ideal, the above proof of Theorem 4.19 shows that $M = M_{(p), \alpha'} \cap R$, where α' is a root of $q(X)$. Hence, such M 's are finitely many. $\square$

4.2.3. Unique factorization domains. In this section we completely describe the UFDs between $\mathbb{Z}_{(p)}[X]$ and $\mathbb{Q}[X]$, where $p \in \mathbb{Z}$ is a fixed prime. It is known that an integral domain is a UFD if and only if it is a Krull domain with trivial divisor class group [13, Proposition 6.1].

The following lemma follows from [29, Proposition 3.2 & Proposition 3.7], but for the sake of the reader we give a self-contained proof.

Lemma 4.21. *Suppose $\alpha \in \overline{\mathbb{Q}_p}$ is unramified over $\mathbb{Q}_p$ and let $q_\alpha \in \mathbb{Q}_p[X]$ be its minimal polynomial. Then, $v_{p,\alpha,\delta}(q_\alpha) = k\delta + \gamma$, for some $\gamma \in \mathbb{Z}$ and $k = \#(\Omega_\alpha \cap \bar{B}_p(\alpha, \delta))$, where Ω_α is the set of conjugates of α over $\mathbb{Q}_p$.*

Proof. We remark that $\Omega_\alpha \subset \mathbb{Q}_p(\alpha)$ because $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ is a Galois extension. Note also that $q_\alpha(X) = \prod_{\alpha' \in \Omega_\alpha} (X - \alpha')$ over $\mathbb{Q}_p(\alpha)$. For each $\alpha' \in \Omega_\alpha$, we have

$$v_{p,\alpha,\delta}(X - \alpha') = \min\{\delta, v_p(\alpha - \alpha')\}$$

so that

$$v_{p,\alpha,\delta}(q_\alpha) = \sum_{\alpha' \in \Omega_\alpha \cap \bar{B}_p(\alpha, \delta)} \delta + \sum_{\alpha' \in \Omega_\alpha \setminus \bar{B}_p(\alpha, \delta)} v_p(\alpha - \alpha') = k\delta + \gamma$$

where $\gamma = \sum_{\alpha' \in \Omega_\alpha \setminus \bar{B}_p(\alpha, \delta)} v_p(\alpha - \alpha')$. Let $O_{\mathbb{Q}_p(\alpha)}$ be the ring of integers of $\mathbb{Q}_p(\alpha)$, so $O_{\mathbb{Q}_p(\alpha)}$ is a valuation domain. Then, as α is unramified, $O_{\mathbb{Q}_p(\alpha)}$ and $\mathbb{Z}_p$ have the same value group, and thus we have $\gamma \in \mathbb{Z}$. More precisely, $\gamma = v_p(h(\alpha))$, where $h(X) = \prod_{\alpha' \in \Omega_\alpha \setminus \bar{B}_p(\alpha, \delta)} (X - \alpha')$. $\square$

Next, we show that the cardinality of the set of conjugates of α over $\mathbb{Q}_p$ in $\bar{B}_p(\alpha, \delta)$, namely, the quantity $\#(\Omega_\alpha \cap \bar{B}_p(\alpha, \delta))$ is 1 when α is unramified over $\mathbb{Q}_p$ and (α, δ) is a minimal pair.

Proposition 4.22. *Let $(\alpha, \delta) \in \overline{\mathbb{Q}_p} \times \mathbb{Q}$ be a minimal pair such that α is unramified over $\mathbb{Q}_p$. Then $\#(\Omega_\alpha \cap \bar{B}_p(\alpha, \delta)) = 1$. In particular, $\gamma = v_{p,\alpha,\delta}(q_\alpha) = \delta + \lambda$, for some $\lambda \in \mathbb{Z}$ and we therefore have:*

$$e(\gamma \mid \mathbb{Q}_p(\alpha)) = e(\delta \mid \mathbb{Q}_p)$$

Proof. Suppose first that $\alpha \in \mathbb{Q}_p$, so that $q_\alpha(X) = X - \alpha$ and $\Omega_\alpha = \{\alpha\}$. Then $v_{p,\alpha,\delta}(q_\alpha) = \delta$ and the statement is true in this case.

Suppose now that $\alpha \notin \mathbb{Q}_p$. We consider the following constants:

$$\begin{aligned} \delta_{\mathbb{Q}_p}(\alpha) &= \sup\{v_p(\alpha - c) \mid c \in \overline{\mathbb{Q}_p}, [\mathbb{Q}_p(c) : \mathbb{Q}_p] < [\mathbb{Q}_p(\alpha) : \mathbb{Q}_p]\} \\ \omega_{\mathbb{Q}_p}(\alpha) &= \sup\{v_p(\alpha - \alpha') \mid \alpha' \in \Omega_\alpha \setminus \{\alpha\}\} \end{aligned}$$

which are both well-defined because $\alpha \notin \mathbb{Q}_p$. It is known that $\delta_{\mathbb{Q}_p}(\alpha)$ is a maximum (see for example [28, Lemma 1.2]) and so in general we have the inequality $\delta_{\mathbb{Q}_p}(\alpha) \leq \omega_{\mathbb{Q}_p}(\alpha)$ (which follows by the classical Krasner's lemma). Since $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ is unramified, by [22, Theorem 1.3] we have the equality $\delta_{\mathbb{Q}_p}(\alpha) = \omega_{\mathbb{Q}_p}(\alpha)$. Moreover, from the definitions it follows that

(α, δ) is a minimal pair if and only if $\delta > \delta_{\mathbb{Q}_p(\alpha)}$. Now, if $\beta \in \bar{B}_p(\alpha, \delta)$, we have

$$v_p(\alpha - \beta) \geq \delta > \delta_{\mathbb{Q}_p(\alpha)} = \omega_{\mathbb{Q}_p}(\alpha)$$

so, the only conjugate of α in $\bar{B}_p(\alpha, \delta)$ is α itself, that is, $\#(\Omega_\alpha \cap \bar{B}_p(\alpha, \delta)) = 1$.

The last claim now follows by Lemma 4.21 and the displayed equation is clear from the fact that $\lambda \in \mathbb{Z}$. $\square$

The next theorem characterizes when a DVR of $\mathbb{Q}(X)$ extending $\mathbb{Z}_{(p)}$ is unramified over $\mathbb{Z}_{(p)}$.

Theorem 4.23. *Let W be a DVR of $\mathbb{Q}(X)$ extending $\mathbb{Z}_{(p)}$ containing X . Then W is unramified over $\mathbb{Z}_{(p)}$ if and only if there exists a minimal pair $(\alpha, \delta) \in \mathbb{O}_p^{br} \times (\mathbb{Z}_{\geq 0} \cup \{\infty\})$ such that $W = \mathbb{Z}_{(p), \alpha, \delta}$ and $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ is unramified.*

Proof. By Theorem 3.6, there exists $(\alpha, \delta) \in \mathbb{C}_p^{br} \times (\mathbb{Q} \cup \{\infty\})$ such that $W = \mathbb{Z}_{(p), \alpha, \delta}$.

Suppose first that W is residually algebraic over $\mathbb{Z}_{(p)}$, so, by the same Theorem $\delta = \infty$ and α is transcendental over $\mathbb{Q}$. In this case, if $\alpha \in \overline{\mathbb{Q}_p}$, then by [25, Proposition 2.2] we have $e(W \mid \mathbb{Z}_{(p)}) = e(\mathbb{Q}_p(\alpha) \mid \mathbb{Q}_p)$. If instead $\alpha \in \mathbb{C}_p \setminus \overline{\mathbb{Q}_p}$, then $e(W \mid \mathbb{Z}_{(p)}) = e(\mathbb{Z}_{p, \alpha, \delta} \mid \mathbb{Z}_p) = e(\mathbb{Q}_p(\alpha) \mid \mathbb{Q}_p)$, where the first equality follows from [28, Proposition 2.24] and the second equality from [28, Proposition 2.14]. In either case, the claim follows.

Suppose now that W is residually transcendental over $\mathbb{Z}_{(p)}$, so in particular, $\delta \in \mathbb{Q}$ by the aforementioned Theorem, and we may assume that $\alpha \in \overline{\mathbb{Q}_p}$. By [2, Theorem 2.1], $e(W \mid \mathbb{Z}_{(p)}) = 1$ if and only if the following condition holds:

there exists a minimal pair $(\alpha, \delta) \in \overline{\mathbb{Q}_p} \times \mathbb{Q}$ of definition of W such that $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ is unramified and $w(q_\alpha) \in \Gamma_{\mathbb{Q}_p(\alpha)}$ where q_α is the minimal polynomial of α over $\mathbb{Q}_p$.

By Lemma 4.21, we have $w(q_\alpha) = v_{p, \alpha, \delta}(q_\alpha) = k\delta + \lambda$, where $k = \#(\Omega_\alpha \cap \bar{B}_p(\alpha, \delta))$ and $\lambda \in \Gamma_{\mathbb{Q}_p(\alpha)} = \Gamma_{\mathbb{Q}_p} = \mathbb{Z}$; in particular, we have $w(q_\alpha) \in \Gamma_{\mathbb{Q}_p(\alpha)} \Leftrightarrow k\delta \in \mathbb{Z}$.

If $\alpha \in \mathbb{Q}_p$, then $q_\alpha(X) = X - \alpha$, so $v_{p, \alpha, \delta}(q_\alpha) = \delta$ which is therefore an integer. Suppose that $\alpha \notin \mathbb{Q}_p$. Since (α, δ) is a minimal pair, $\delta > 0$. By Proposition 4.22 we have $k = 1$, so that above displayed condition is equivalent to the condition of the main statement of the theorem. $\square$

Theorem 4.24. *Let R be a ring between $\mathbb{Z}_{(p)}[X]$ and $\mathbb{Q}[X]$. Then R is a UFD if and only if either $R = \mathbb{Q}[X]$ or $R = W \cap \mathbb{Q}[X]$ where W is a DVR of $\mathbb{Q}(X)$ extending $\mathbb{Z}_{(p)}$ unramified over $\mathbb{Z}_{(p)}$. More precisely, the last condition holds if and only if there exists $(\alpha, \delta) \in \mathbb{O}_p^{br} \times (\mathbb{Z}_{\geq 0} \cup \{\infty\})$ such that α is unramified over $\mathbb{Q}_p$ and in this case $R = \text{Int}_{\mathbb{Q}}(\bar{B}_p(\alpha, \delta), \mathbb{O}_p)$.*

Proof. A UFD is just a Krull domain with trivial divisor class group [13, Proposition 6.1]. So, if R is a UFD, then, by Theorem 4.15 we have $R = \bigcap_{i=1}^m \mathbb{Z}_{(p), \alpha_i, \delta_i} \cap \mathbb{Q}[X]$, for some $(\alpha_i, \delta_i) \in \mathbb{O}_p^{\text{br}} \times (\mathbb{Q}_{\geq 0} \cup \{\infty\})$, for $i = 1, \dots, n$. By items (4) and (5) of Theorem 4.10, either $R = \mathbb{Q}[X]$ or $R = \mathbb{Z}_{(p), \alpha, \delta} \cap \mathbb{Q}[X]$ and $\mathbb{Z}_{(p), \alpha, \delta}$ is unramified over $\mathbb{Z}_{(p), \alpha, \delta}$. Conversely, if R has the stated form, then by Theorem 4.10, R is a Krull domain with trivial class group, hence, a Krull domain.

The proof of the last claim follows from Theorem 4.23. $\square$

Example 4.25. Let R be a UFD between $\mathbb{Z}_{(p)}[X]$ and $\mathbb{Q}[X]$. We remark that R is equal to $\mathbb{Q}[X]$ precisely when p is a unit of R , otherwise $R = W \cap \mathbb{Q}[X]$ for some DVR W of $\mathbb{Q}(X)$ extending $\mathbb{Z}_{(p)}$, such that W is unramified over $\mathbb{Z}_{(p)}$. Note that $R = \mathbb{Z}_{(p)}[X]$ if and only if $W = \mathbb{Z}_{(p), 0, 0}$.

4.3. Global case. In this section, we globalize the results of Section 4.2. Namely, we consider a subset $\Lambda \subseteq \mathbb{P}$, $W_{p,j}$ a DVR of $\mathbb{Q}(X)$ with $X \in W_{p,j}$ and $W_{p,j} \cap \mathbb{Q} = \mathbb{Z}_{(p)}$, and $R = \bigcap_{p \in \Lambda} \left(\bigcap_{j=1}^{m_p} (W_{p,j} \cap \mathbb{Q}[X]) \right)$.

4.3.1. When a polynomial Krull domain is Dedekind. Let R be a polynomial Krull domain. The next proposition shows that if all the unitary DVRs of the defining family of R are residually algebraic extensions of $\mathbb{Z}_{(p)}$, for $p \in \mathbb{P}$, then R is a Dedekind domain. We first recall the following definition given in [27, 28].

Definition 4.26. Let $\underline{E} = \prod_{p \in \mathbb{P}} E_p \subseteq \mathbb{O} = \prod_{p \in \mathbb{P}} \mathbb{O}_p$ be a subset. We say that $\underline{E}$ is *polynomially factorizable* if, for each $g \in \mathbb{Z}[X]$ and $\alpha = (\alpha_p) \in \underline{E}$, there exist $n, d \in \mathbb{Z}$, $n, d \geq 1$ such that $\frac{g(\alpha)^n}{d}$ is a unit of $\mathbb{O}$, that is, $v_p(\frac{g(\alpha)^n}{d}) = 0$ for all $p \in \mathbb{P}$.

If every $\alpha_p \in E_p$ is transcendental over $\mathbb{Q}$ for every $p \in \mathbb{P}$, [27, Lemma 2.12] shows that a set $\underline{E}$ as in the above definition is polynomially factorizable if and only if, for each $g \in \mathbb{Z}[X]$, the following set is finite:

$$\mathbb{P}_{g, \underline{E}} = \{p \in \mathbb{P} \mid \exists \alpha_p \in E_p \text{ such that } v_p(g(\alpha_p)) > 0\}.$$

Clearly, we have

$$\mathbb{P}_{g, \underline{E}} = \{p \in \mathbb{P} \mid \exists \alpha_p \in E_p \text{ such that } g \in \mathbb{Z}[X] \cap M_{(p), \alpha_p}\},$$

where $M_{(p), \alpha_p} = M_{(p), \alpha_p, \infty}$ is the maximal ideal of $\mathbb{Z}_{(p), \alpha_p} = \mathbb{Z}_{(p), \alpha_p, \infty}$.

Proposition 4.27. *Let R be a Krull domain between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$. Then the following statements are equivalent.*

- (1) R is a Dedekind domain.
- (2) Each unitary DVR W of the defining family of R is a residually algebraic extension of $W \cap \mathbb{Q}$.

- (3) $R = \text{Int}_{\mathbb{Q}}(\underline{E}, \mathbb{O})$, for some subset $\underline{E} = \prod_{p \in \mathbb{P}} E_p \subset \mathbb{O} = \prod_{p \in \mathbb{P}} \mathbb{O}_p$ such that $E_p \subset \mathbb{O}_p^{\text{br}}$ is a finite set of transcendental elements over $\mathbb{Q}$ for each $p \in \mathbb{P}$.

In this case, $\underline{E}$ is polynomially factorizable.

Proof. (1) $\Rightarrow$ (2) This is an immediate consequence of Theorem 4.10, (3).

(2) $\Rightarrow$ (3) Let $\{W_i\}_{i \in I}$ be the proper subset of the defining family of DVRs of R which are extensions of $\mathbb{Z}_{(p)}$ for some $p \in \mathbb{P}$. By Theorem 3.1, if W_i is a residually algebraic extension of $\mathbb{Z}_{(p)}$, then there exists $\alpha_p \in \mathbb{O}_p^{\text{br}}$, transcendental over $\mathbb{Q}$, such that $W_i = \mathbb{Z}_{(p), \alpha_p} = \mathbb{Z}_{(p), \alpha_p, \infty}$. For each $p \in \mathbb{P}$, we denote by E_p the subset formed by these $\alpha_p \in \mathbb{O}_p^{\text{br}}$. Then the defining family of DVRs of R is equal to:

$$\{\mathbb{Z}_{(p), \alpha_p} \mid p \in \mathbb{P}, \alpha_p \in E_p\} \cup \{\mathbb{Q}[X]_{(g)} \mid g \in \mathbb{Q}[X], \text{ irreducible}\}$$

In particular, we have

$$R = \bigcap_{i \in I} W_i \cap \mathbb{Q}[X] = \bigcap_{p \in \mathbb{P}} \bigcap_{\alpha_p \in E_p} \mathbb{Z}_{(p), \alpha_p} \cap \mathbb{Q}[X] = \bigcap_{p \in \mathbb{P}} \text{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p)$$

where the last equality follows from Remark 3.12. Note that we also have

$$\bigcap_{p \in \mathbb{P}} \text{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p) = \text{Int}_{\mathbb{Q}}(\underline{E}, \mathbb{O}) = \{f \in \mathbb{Q}[X] \mid f(\alpha) \in \mathbb{O}, \forall \alpha \in \underline{E}\}$$

where $\underline{E} = \prod_p E_p$, $\mathbb{O} = \prod_p \mathbb{O}_p$ and $f(\alpha) = (f(\alpha_p))_p$ if $\alpha = (\alpha_p)_p \in \underline{E}$. Finally,

let $p \in \mathbb{P}$. Then, since $p \in R$, p is contained in finitely many centers of the defining family of DVRs of R . This implies that E_p is finite.

(3) $\Rightarrow$ (1) Let M be a maximal ideal of R . If $M \cap \mathbb{Z} = (0)$, then $\text{ht} M = 1$, so assume that $M \cap \mathbb{Z} = p\mathbb{Z}$ for some $p \in \mathbb{P}$. Note that $R = \text{Int}_{\mathbb{Q}}(\underline{E}, \mathbb{O}) = \bigcap_{p \in \mathbb{P}} \bigcap_{\alpha_p \in E_p} \mathbb{Z}_{(p), \alpha_p} \cap \mathbb{Q}[X]$, so if $S = \mathbb{Z} \setminus p\mathbb{Z}$, then $R_S = \bigcap_{\alpha_p \in E_p} \mathbb{Z}_{(p), \alpha_p} \cap \mathbb{Q}[X]$ by Lemma 4.3 and $\bigcap_{\alpha_p \in E_p} \mathbb{Z}_{(p), \alpha_p} \cap \mathbb{Q}[X]$ is a Dedekind domain by Theorem 4.15.

Hence, $\text{ht} M = \text{ht}(MR_S) = 1$. Thus, R is a one-dimensional Krull domain, therefore R is a Dedekind domain.

Now for the proof of “In this case”, let $f \in \mathbb{Z}[X] \subset R$. Since R is a Krull domain, f is contained in only finitely many centers of the valuation domains of the defining family for R (clearly, f is contained in finitely many centers of the non-unitary valuation domains $\mathbb{Q}[X]_{(g)}$, which are the non-unitary prime ideals of R). This precisely means that the set $\mathbb{P}_{g, \underline{E}} = \{p \in \mathbb{P} \mid \exists \alpha_p \in E_p, v_p(f(\alpha_p)) > 0\}$ is finite. Hence, $\underline{E}$ is polynomially factorizable by [27, Lemma 2.12]. $\square$

In particular, a polynomial Krull domain which is not Dedekind must have an essential valuation overring which is a residually transcendental extension of $\mathbb{Z}_{(p)}$, for some $p \in \mathbb{P}$.

Proposition 4.28. *Let R be a polynomial Krull domain. Then $R = R_1 \cap R_2$, where R_1 is a Dedekind domain such that $\mathbb{Z}[X] \subset R_1 \subseteq \mathbb{Q}[X]$, and R_2 is either a non-Prüfer Krull domain such that $\mathbb{Z}[X] \subseteq R_2 \subset \mathbb{Q}[X]$ and Q is not maximal for all $Q \in X^1(R_2)$ with $Q \cap \mathbb{Z} \neq (0)$ or $R_2 = \mathbb{Q}[X]$.*

Proof. Let $\mathcal{W}(R) = \{W_i\}_{i \in I}$ be the subset of the defining family of DVRs for R which are unitary valuation domains. Note that $\mathcal{W}(R) = \emptyset$ is equivalent to saying that $R = \mathbb{Q}[X]$; in this case we have $R_1 = R_2 = \mathbb{Q}[X]$. Suppose now that $\mathcal{W}(R) \neq \emptyset$. We partition $\mathcal{W}(R)$ into the subset $\mathcal{W}_{\text{ra}}(R)$ of residually algebraic extensions of $\mathbb{Z}_{(p)}$ for some $p \in \mathbb{P}$ and the subset $\mathcal{W}_{\text{rt}}(R)$ of residually transcendental extensions of $\mathbb{Z}_{(p)}$ for some $p \in \mathbb{P}$.

We set:

$$R_1 = \bigcap_{W_i \in \mathcal{W}_{\text{ra}}(R)} W_i \cap \mathbb{Q}[X] \quad \text{and} \quad R_2 = \bigcap_{W_i \in \mathcal{W}_{\text{rt}}(R)} W_i \cap \mathbb{Q}[X].$$

Then $R = R_1 \cap R_2$. Clearly, $R_1 = \mathbb{Q}[X]$ if and only if $\mathcal{W}_{\text{ra}}(R) = \emptyset$ and similarly $R_2 = \mathbb{Q}[X]$ if and only if $\mathcal{W}_{\text{rt}}(R) = \emptyset$. Note that since R_1 and R_2 are both subintersections of the Krull domain R , they both are Krull domains. Moreover, by Proposition 4.27, R_1 is Dedekind, while if $\mathcal{W}_{\text{rt}}(R) \neq \emptyset$, then R_2 is not Prüfer, since every $W_i \in \mathcal{W}_{\text{rt}}(R)$ is residually transcendental over $W_i \cap \mathbb{Q}$ (see [26, Theorem 3.14]).

Now, suppose that $R_2 \subset \mathbb{Q}[X]$. Let $Q \in X^1(R_2)$ be such that $Q \cap \mathbb{Z} \neq (0)$, so $Q \cap \mathbb{Q} = p\mathbb{Z}_{(p)}$ for some $p \in \mathbb{P}$. Then $(R_2)_Q = W_k$ for some $W_k \in \mathcal{W}_{\text{rt}}(R)$ by [10, Corollary 2.3(i)]. It is clear that if S is the multiplicative subset of $\mathbb{Z}$ generated by $\mathbb{P} - \{p\}$, then $(R_2)_S$ is a Krull domain between $\mathbb{Z}_{(p)}[X]$ and $\mathbb{Q}[X]$. Note that $(R_2)_S = \bigcap_{W_i \in \mathcal{W}_{\text{rt}}(R)} (W_i)_S \cap \mathbb{Q}[X]$ and $(W_k)_S = W_k$, so

Q_S is not a maximal ideal of $(R_2)_S$ by Theorem 4.10 (3). Thus Q is not a maximal ideal of R_2 . $\square$

In Proposition 4.28, $R_1 = \mathbb{Q}[X]$ if R has no valuation overrings which are residually algebraic extensions of $\mathbb{Z}_{(p)}$ for all $p \in \mathbb{P}$, and $R_2 = \mathbb{Q}[X]$ if R is a Dedekind domain (or, equivalently, R has no valuation overrings which are residually transcendental extensions of $\mathbb{Z}_{(p)}$ for all $p \in \mathbb{P}$). Therefore, the objective is now to characterize when, for a given family $\{W_i\}_{i \in I}$ of residually transcendental extensions of $\mathbb{Z}_{(p)}$, $p \in \mathbb{P}$, the ring $\bigcap_{i \in I} W_i \cap \mathbb{Q}[X]$ has finite character, and thus is a Krull domain.

4.3.2. Centers of monomial valuations on $\mathbb{Z}[X]$. Let $p \in \mathbb{P}$ be fixed. Given $(\alpha, \delta) \in \mathbb{C}_p \times (\mathbb{Q} \cup \{\infty\})$, we denote the maximal ideal of $\mathbb{Z}_{(p), \alpha, \delta}$ by $M_{(p), \alpha, \delta}$. By Theorem 3.6, $\mathbb{Z}_{(p), \alpha, \delta}$ is an overring of $\mathbb{Z}_{(p)}[X]$ if and only if $(\alpha, \delta) \in \mathbb{O}_p \times (\mathbb{Q}_{\geq 0} \cup \{\infty\})$. In this section, we restrict to this case and describe the center of $\mathbb{Z}_{(p), \alpha, \delta}$ on $\mathbb{Z}_{(p)}[X]$, namely, $M_{(p), \alpha, \delta} \cap (\mathbb{Z}_{(p)}[X])$.

Let $\mathbb{M}_p$ be the maximal ideal of $\mathbb{O}_p$. We set $\text{Int}(\bar{B}_p(\alpha, \delta), \mathbb{M}_p) = \{f \in \mathbb{C}_p[X] \mid f(\bar{B}_p(\alpha, \delta)) \subseteq \mathbb{M}_p\}$. It is easy to see that $\text{Int}(\bar{B}_p(\alpha, \delta), \mathbb{M}_p)$ is an ideal

of $\text{Int}(\bar{B}_p(\alpha, \delta), \mathbb{O}_p)$. The following lemma shows that this ideal is precisely the center of $\mathbb{O}_{p, \alpha, \delta}$ on $\mathbb{O}_{p, \alpha, \delta} \cap \mathbb{C}_p[X] = \text{Int}(\bar{B}_p(\alpha, \delta), \mathbb{O}_p)$ (see (3.13)).

Lemma 4.29. *Let $p \in \mathbb{P}$, $(\alpha, \delta) \in \mathbb{C}_p \times (\mathbb{Q} \cup \{\infty\})$ and $\mathbb{M}_{p, \alpha, \delta}$ be the maximal ideal of $\mathbb{O}_{p, \alpha, \delta}$. Then, we have*

$$\mathbb{M}_{p, \alpha, \delta} \cap (\mathbb{O}_{p, \alpha, \delta} \cap \mathbb{C}_p[X]) = \text{Int}(\bar{B}_p(\alpha, \delta), \mathbb{M}_p).$$

Proof. The statement clearly holds if $\delta = \infty$. Suppose then that $\delta < \infty$.

($\subseteq$) Let $f \in \mathbb{M}_{p, \alpha, \delta} \cap (\mathbb{O}_{p, \alpha, \delta} \cap \mathbb{C}_p[X])$. Then, if $f(X) = \sum_k a_k(X - \alpha)^k$, we have $v_p(a_k) + k\delta > 0$ for each k . In particular, if $\alpha' \in \mathbb{C}_p$ is such that $v_p(\alpha - \alpha') \geq \delta$, we have $f(\alpha') = \sum_k a_k(\alpha' - \alpha)^k$, so that $v_p(f(\alpha')) \geq \min_k \{v_p(a_k) + k\delta\} > 0$. Hence, $f(\bar{B}_p(\alpha, \delta)) \subseteq \mathbb{M}_p$.

($\supseteq$) Let $f \in \text{Int}(\bar{B}_p(\alpha, \delta), \mathbb{M}_p) \subset \text{Int}(\bar{B}_p(\alpha, \delta), \mathbb{O}_p)$. Let $E = \{\alpha_n\}_{n \in \mathbb{N}} \subset \bar{B}_p(\alpha, \delta)$ be a pseudo-stationary sequence such that $\delta_E = \delta$ and let $v_{p, E}$ be the associated valuation (see [29]). By [29, Proposition 3.7], $v_{p, \alpha, \delta} = v_{p, E}$ and by [29, Theorem 3.4], $f(X)$ is in the maximal ideal of $\mathbb{O}_{p, E} = \mathbb{O}_{p, \alpha, \delta}$, because by assumption $v_p(f(\alpha_n)) > 0$ for every $n \in \mathbb{N}$; thus, $f \in \mathbb{M}_{p, \alpha, \delta}$. $\square$

For each $p \in \mathbb{P}$, let $\mathbb{F}_p$ be the finite field of p elements, i.e., $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and $\bar{f}$ be the image of $f \in \mathbb{Z}[X]$ in $\mathbb{F}_p[X]$. Let $\text{Spec}(\mathbb{Z}[X])$ (resp., $\text{Max}(\mathbb{Z}[X])$) be the set of prime ideals (resp., maximal ideals) of $\mathbb{Z}[X]$. Note that $\dim(\mathbb{Z}[X]) = 2$ and each maximal ideal of $\mathbb{Z}[X]$ has height two, so

$$\text{Spec}(\mathbb{Z}[X]) = \{0\} \cup X^1(\mathbb{Z}[X]) \cup \text{Max}(\mathbb{Z}[X]) \text{ and } X^1(\mathbb{Z}[X]) \cap \text{Max}(\mathbb{Z}[X]) = \emptyset.$$

It is well known and easy to see that

- (a) $X^1(\mathbb{Z}[X]) = \{p\mathbb{Z}[X] \mid p \in \mathbb{P}\} \cup \{f\mathbb{Z}[X] \mid f \in \mathbb{Z}[X] \text{ and } f \text{ is irreducible in } \mathbb{Z}[X]\}$
- (b) $\text{Max}(\mathbb{Z}[X]) = \{(p, f) \mid p \in \mathbb{P}, f \in \mathbb{Z}[X] \text{ and } \bar{f}\mathbb{F}_p[X] \text{ is a prime ideal}\}.$

In particular, for each $p \in \mathbb{P}$,

$$\begin{aligned} \text{Spec}(\mathbb{Z}_{(p)}[X]) &= \{0\} \cup \{p\mathbb{Z}_{(p)}[X]\} \\ &\cup \{f\mathbb{Z}_{(p)}[X] \mid f \in \mathbb{Z}_{(p)}[X] \text{ and } f \text{ is irreducible in } \mathbb{Z}_{(p)}[X]\} \\ &\cup \{(p, f) \mid \bar{f}\mathbb{F}_p[X] \text{ is a prime ideal}\}. \end{aligned}$$

Recall that if $\alpha \in \mathbb{O}_p$, then $\bar{\alpha}$ is the image of α modulo $\mathbb{M}_p$ and $g_\alpha(X) \in \mathbb{Z}_{(p)}[X]$ denotes a monic polynomial such that $\overline{g_\alpha(X)}$ is the minimal polynomial over $\bar{\alpha}$ over $\mathbb{F}_p$.

Proposition 4.30. *Let $p \in \mathbb{P}$ and $(\alpha, \delta) \in \mathbb{O}_p \times (\mathbb{Q}_{\geq 0} \cup \{\infty\})$. Then*

$$\mathbb{Z}_{(p)}[X] \cap M_{(p), \alpha, \delta} = \begin{cases} (p), & \text{if } \delta = 0 \\ (p, g_\alpha(X)), & \text{if } \delta > 0 \end{cases}$$

Proof. Suppose first that $\delta = 0$. Then $\mathbb{Z}_{(p), \alpha, 0} = \mathbb{Z}_{(p), 0, 0} = \mathbb{Z}_{(p)}[X]_{p\mathbb{Z}_{(p)}[X]}$ (i.e., the trivial Gaussian extension). In particular, $\mathbb{Z}_{(p)}[X] = \mathbb{Z}_{(p), 0, 0} \cap \mathbb{Q}[X]$, so, the center of $M_{(p), \alpha, 0} = M_{(p), 0, 0}$ on $\mathbb{Z}_{(p)}[X]$ is equal to $p\mathbb{Z}_{(p)}[X]$.

Suppose now that $\delta = \infty$. Since $(p, g_\alpha(X))$ is a maximal ideal of $\mathbb{Z}_{(p)}[X]$ and $g_\alpha \in M_{(p), \alpha, \infty}$, $(p, g_\alpha(X))$ is the center of $\mathbb{Z}_{(p), \alpha, \infty}$ on $\mathbb{Z}_{(p)}[X]$.

Finally, suppose that $0 < \delta < \infty$. By Lemma 4.29,

$$M_{(p), \alpha, \delta} \cap (\mathbb{Z}_{(p), \alpha, \delta} \cap \mathbb{Q}[X]) = \text{Int}_{\mathbb{Q}}(\bar{B}_p(\alpha, \delta), \mathbb{M}_p),$$

so $\mathbb{Z}_{(p)}[X] \cap M_{(p), \alpha, \delta} = \{f \in \mathbb{Z}_{(p)}[X] \mid f(\bar{B}_p(\alpha, \delta)) \subseteq \mathbb{M}_p\}$. Hence, in order to show that this last intersection is equal to $(p, g_\alpha(X))$, it is sufficient to show that g_α maps $\bar{B}_p(\alpha, \delta)$ into $\mathbb{M}_p$. If $\alpha' \in \mathbb{C}_p$ is such that $v(\alpha' - \alpha) \geq \delta > 0$, then $g_\alpha(\alpha') \equiv g_\alpha(\alpha) \equiv 0 \pmod{\mathbb{M}_p}$, which is what we wanted to prove. $\square$

Remark 4.31. Clearly, if $\bar{B}_p(\alpha, \delta) \subset \mathbb{O}_p$ with $\delta > 0$, then for all $\alpha' \in \bar{B}_p(\alpha, \delta)$, we have $\bar{\alpha} = \bar{\alpha}'$, so $\bar{g}_\alpha = \bar{g}_{\alpha'}$. Hence, for $\delta > 0$, the center of $\mathbb{Z}_{(p), \alpha, \delta}$ on $\mathbb{Z}_{(p)}[X]$ does not depend on the particular center of the ball $\bar{B}(\alpha, \delta)$. Most importantly, by Lemma 4.30, given $\alpha \in \mathbb{O}_p$, the center of $\mathbb{Z}_{(p), \alpha, \delta}$ on $\mathbb{Z}_{(p)}[X]$ is the same maximal ideal for all $\delta \in \mathbb{Q}_{>0} \cup \{\infty\}$.

Putting these two remarks together, for each $\delta \in \mathbb{Q}, \delta > 0$, we have that

$$M_{(p), \alpha, \delta} \cap \mathbb{Z}[X] = M_{(p), \alpha', \infty} \cap \mathbb{Z}[X], \quad \forall \alpha' \in \bar{B}_p(\alpha, \delta).$$

In particular, for $\delta > 0$ and $g \in \mathbb{Z}[X]$, we have that $g \in M_{(p), \alpha, \delta}$ if and only if $g \in M_{(p), \alpha', \infty}$ for some (hence, all) $\alpha' \in \bar{B}_p(\alpha, \delta)$.

Corollary 4.32. *Let $p \in \mathbb{P}$ and $(\alpha_i, \delta_i) \in \mathbb{O}_p \times (\mathbb{Q}_{\geq 0} \cup \{\infty\})$, for $i = 1, 2$. Then $\mathbb{Z}_{(p)}[X] \cap M_{(p), \alpha_1, \delta_1} = \mathbb{Z}_{(p)}[X] \cap M_{(p), \alpha_2, \delta_2}$ if and only if one of the following conditions holds:*

- (a) $\delta_1 = \delta_2 = 0$;
- (b) $\delta_i \neq 0$, for $i = 1, 2$, and $\bar{\alpha}_1, \bar{\alpha}_2$ are conjugate over $\mathbb{F}_p$. Equivalently, $v_p(\sigma(\alpha_1) - \alpha_2) > 0$ for some $\sigma \in G_p = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$.

Proof. $(\Rightarrow)$ Suppose first that $\mathbb{Z}_{(p), \alpha_1, \delta_1}, \mathbb{Z}_{(p), \alpha_2, \delta_2}$ have the same center on $\mathbb{Z}_{(p)}[X]$. By Proposition 4.30, we have either $\delta_1 = \delta_2 = 0$ or $\delta_i \neq 0$ for $i = 1, 2$. In the last case, we have $\mathbb{Z}_{(p)}[X] \cap M_{(p), \alpha_i, \delta_i} = (p, g_{\alpha_i}(X))$, where $g_{\alpha_i} \in \mathbb{Z}_{(p)}[X]$ is a monic polynomial such that $\bar{g}_{\alpha_i}$ is the minimal polynomial of $\bar{\alpha}_i$ over $\mathbb{F}_p$, for $i = 1, 2$. Then, these prime ideals are the same if and only if $\bar{g}_{\alpha_1} = \bar{g}_{\alpha_2}$, that is, $\bar{\alpha}_1, \bar{\alpha}_2$ are conjugate over $\mathbb{F}_p$. Note that this last condition amounts to saying that there exists some $\bar{\sigma} \in \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ such that $\bar{\sigma}(\bar{\alpha}_1) = \bar{\alpha}_2$. Since $\bar{\sigma}(\bar{\alpha}_1) = \overline{\sigma(\alpha_1)}$ for some $\sigma \in G_p$ [7, Chapt. V, §2.2, Proposition 6(ii)], it follows that $v_p(\sigma(\alpha_1) - \alpha_2) > 0$, as claimed.

$(\Leftarrow)$ If $\delta_1 = \delta_2 = 0$, then $\mathbb{Z}_{(p)}[X] \cap M_{(p), \alpha_i, \delta_i} = (p)$ for $i = 1, 2$. If instead $\delta_i \neq 0$ for $i = 1, 2$ and $\bar{\alpha}_1, \bar{\alpha}_2$ are conjugate over $\mathbb{F}_p$, then by the above arguments we have $\mathbb{Z}_{(p)}[X] \cap M_{(p), \alpha_1, \delta_1} = \mathbb{Z}_{(p)}[X] \cap M_{(p), \alpha_2, \delta_2}$. $\square$

In Corollary 4.32, if $\delta_i \neq 0$ for $i = 1, 2$, we have the following three possibilities: (i) α_1, α_2 are conjugate by the action of G_p and $\delta_1 = \delta_2$ and so $\mathbb{Z}_{(p), \alpha_1, \delta_1} = \mathbb{Z}_{(p), \alpha_2, \delta_2}$ (and so in particular they have the same center on $\mathbb{Z}_{(p)}[X]$), (ii) α_1, α_2 are not conjugate over $\mathbb{Q}_p$ or (iii) $\delta_1 \neq \delta_2$ so that $\mathbb{Z}_{(p), \alpha_1, \delta_1} \neq \mathbb{Z}_{(p), \alpha_2, \delta_2}$ but $\bar{\alpha}_1, \bar{\alpha}_2$ are conjugate over $\mathbb{F}_p$ so the corresponding

valuation domains have the same center on $\mathbb{Z}_{(p)}[X]$. For example, if $\alpha_1, \alpha_2 \in \mathbb{O}_p$ are such that $v_p(\alpha_1 - \alpha_2) > 0$ (that is, they belong to the same ball contained in $\mathbb{O}_p$ of strictly positive radius), then $\overline{\alpha_1} = \overline{\alpha_2}$.

4.3.3. Finite character. A Polynomial Krull domain R must be of the form $R = \bigcap_{p \in \Lambda} \left(\bigcap_{j=1}^{m_p} (W_{p,j} \cap \mathbb{Q}[X]) \right)$, for some $\Lambda \subseteq \mathbb{P}$ and DVRs $W_{p,j}$, for each $p \in \Lambda$ and $j \in \{1, \dots, m_p\}$. But, in general, a ring of this form need not be a polynomial Krull domain (see, for example, [10, Theorem 3.5] or Example 4.33).

Example 4.33. For each prime $p \in \mathbb{P}$, let $\delta_p \in \mathbb{Q}$, $\delta_p > 0$, and let

$$R = \bigcap_{p \in \mathbb{P}} \mathbb{Z}_{(p), 0, \delta_p} \cap \mathbb{Q}[X].$$

Then R is not Krull, because X is contained in each center of the residually transcendental extensions $\mathbb{Z}_{(p), 0, \delta_p}$ of $\mathbb{Z}_{(p)}$, for $p \in \mathbb{P}$.

In this section, we determine when such an intersection is a Krull domain.

We consider the case of a subset $E = \prod_{p \in \mathbb{P}} E_p$ of $\mathbb{O}$ such that, for each $p \in \mathbb{P}$, E_p is a finite union of balls in $\mathbb{O}_p^{\text{br}}$:

$$E_p = \bigcup_{j=1}^{m_p} \bar{B}_p(\alpha_{p,j}, \delta_{p,j}), \quad (\alpha_{p,j}, \delta_{p,j}) \in \mathbb{O}_p^{\text{br}} \times (\mathbb{Q}_{\geq 0} \cup \{\infty\}) \quad \text{for } j = 1, \dots, m_p,$$

with the assumption that $\alpha_{p,j}$ is transcendental over $\mathbb{Q}$ for each $p \in \mathbb{P}$ and $j \in \{1, \dots, m_p\}$. If $\delta_{p,j} \neq \infty$ this can always be assumed by a density argument and if $\delta_{p,j} = \infty$ this condition has to be imposed. For each $p \in \mathbb{P}$, we consider the set of centers of the balls in E_p of strictly positive radius given by $C(E_p) = \{\alpha_{p,1}, \dots, \alpha_{p,m_p} \mid \delta_{p,j} > 0, \forall j = 1, \dots, m_p\}$ and set

$$C(\underline{E}) = \prod_{p \in \mathbb{P}} C(E_p)$$

Note that, for each $p \in \mathbb{P}$, the set of the centers of $\{\mathbb{Z}_{(p), \alpha_{p,j}, \delta_{p,j}} \mid j = 1, \dots, m_p\}$ on $\mathbb{Z}[X]$, namely

$$\{\mathbb{Z}[X] \cap M_{(p), \alpha_{p,j}, \delta_{p,j}} \mid j = 1, \dots, m_p\}$$

is finite (possibly, by Corollary 4.32 its cardinality can be strictly smaller than m_p).

We remark that the choice of a center of an ultrametric ball $\bar{B}_p(\alpha, \delta) \subset \mathbb{O}_p$ is completely arbitrary (every element of the ball is a center of the ball itself). However, for our scopes, by Remark 4.31 a possible different choice does not affect the next Theorem 4.34, **provided** the center of each ball is transcendental over $\mathbb{Q}$ (because, in this case, [27, Lemma 2.12] can be applied for the proof of Theorem 4.34). This property can always be assumed if $\delta_{p,j} \in \mathbb{Q}_{\geq 0}$ and it has to be imposed when $\delta_{p,j} = \infty$.

Theorem 4.34. *Let $\Lambda \subseteq \mathbb{P}$ be a subset, $E = \prod_{p \in \Lambda} E_p \subset \mathbb{O}$ where for each $p \in \Lambda$, $E_p = \bigcup_{j=1}^{m_p} \bar{B}_p(\alpha_{p,j}, \delta_{p,j})$ with $(\alpha_{p,j}, \delta_{p,j}) \in \mathbb{O}_p^{br} \times (\mathbb{Q}_{\geq 0} \cup \{\infty\})$, $\alpha_{p,j}$ transcendental over $\mathbb{Q}$, for each $j \in \{1, \dots, m_p\}$ and let $R = \text{Int}_{\mathbb{Q}}(\underline{E}, \mathbb{O})$, where $\underline{E} = \prod_{p \in \Lambda} E_p$, namely:*

$$R = \bigcap_{p \in \Lambda} \bigcap_{j=1}^{m_p} \mathbb{Z}_{(p), \alpha_{p,j}, \delta_{p,j}} \cap \mathbb{Q}[X].$$

Assuming that the above representation of R is irredundant, then the following conditions are equivalent:

- (1) *R is a Krull domain.*
- (2) *For each (nonconstant and irreducible) $g \in \mathbb{Z}[X]$, g is contained in finitely many height one unitary prime ideals of R .*
- (3) *For each $g \in \mathbb{Z}[X]$, the following set is finite:*
 $\mathbb{D}_{g, \underline{E}} = \{p \in \Lambda \mid \exists j \in \{1, \dots, m_p\} \text{ such that } g \in \mathbb{Z}[X] \cap M_{(p), \alpha_{p,j}, \delta_{p,j}}\}.$
- (4) *$C(\underline{E}) = \prod_{p \in \Lambda} \{\alpha_{p,1}, \dots, \alpha_{p,m_p} \mid \forall j = 1, \dots, m_p, \delta_{p,j} > 0\}$ is polynomially factorizable.*

We remark that if for some $p \in \Lambda$ we have $\delta_{p, \alpha_j} = 0$ for $j = 1, \dots, m_p$, then $m_p = 1$ by the assumption that the above representation of R is irredundant. For such a prime p , we have

$$R_{(p)} = \mathbb{Z}_{(p), \alpha_p, 0} \cap \mathbb{Q}[X] = \text{Int}_{\mathbb{Q}}(\mathbb{O}_p) = \mathbb{Z}_{(p)}[X]$$

(see for example (3.14)).

Proof. (1) $\Leftrightarrow$ (2) Note that R is a Krull domain if and only if, for every $f \in R$, $f \neq 0$, f is contained in finitely many centers of the defining family of DVR overrings of R , namely, $\mathbb{Z}_{(p), \alpha_{p,j}, \delta_{p,j}}$, for $p \in \Lambda$ and $j = 1, \dots, m_p$ and $\mathbb{Q}[X]_{(q)}$, for $q \in \mathbb{Q}[X]$ irreducible.

Since every $f \in R$ can be written as $\frac{g}{d}$, for some $g \in \mathbb{Z}[X]$ and $d \in \mathbb{Z}$, $d \neq 0$, it is clear that f is contained in finitely many centers of the defining family of DVR overrings of R if and only if the same holds for g , since the integer d is divisible by finitely many $p \in \Lambda$ and for each $p \in \Lambda$ there are only finitely many DVRs overring of R extending $\mathbb{Z}_{(p)}$. Moreover, g is clearly contained in finitely many nonunitary prime ideals of R (whose height is one by an argument similar to the one of Corollary 4.8), so R is Krull if and only if condition (2) holds (by assumption and by Corollary 4.6, the centers on R of the DVRs $\mathbb{Z}_{(p), \alpha_{p,j}, \delta_{p,j}}$ are of height one).

(2) $\Leftrightarrow$ (3) Since for each $p \in \Lambda$, we only have finitely many unitary DVR overrings of R extending $\mathbb{Z}_{(p)}$ (namely, $\mathbb{Z}_{(p), \alpha_{p,j}, \delta_{p,j}}$, for $j = 1, \dots, m_p$) and their centers on R are of height one by Corollary 4.6, it follows that (2) is equivalent to (3).

(3) $\Leftrightarrow$ (4) Let $\mathbb{P}_0$ be the set of primes in Λ for which there exists $j \in \{1, \dots, m_p\}$ such that $\delta_{p,j} = 0$. If $p \in \mathbb{P}_0$, then $m_p = 1$ by Remark 3.18 and the assumption that the above representation of R is irredundant, and $M_{(p), \alpha_{p,1}, \delta_{p,1}} \cap \mathbb{Z}_{(p)}[X] = (p)$ by Proposition 4.30. Hence, for each nonzero $g \in \mathbb{Z}[X]$, $\mathbb{D}_{g, \underline{E}} \cap \mathbb{P}_0$ is a finite set. By the same aforementioned Proposition, if $p \in \Lambda \setminus \mathbb{P}_0$, we have $M_{(p), \alpha_{p,j}, \delta_{p,j}} \cap \mathbb{Z}_{(p)}[X] = M_{(p), \alpha_{p,j}, \infty} \cap \mathbb{Z}_{(p)}[X]$, so $\mathbb{D}_{g, \underline{E}} \setminus \mathbb{P}_0 = \mathbb{P}_{g, C(\underline{E})}$. Therefore, $\mathbb{D}_{g, \underline{E}}$ is finite if and only if $\mathbb{P}_{g, C(\underline{E})}$ is finite. Since each $\alpha_{p,j}$ is transcendental over $\mathbb{Q}$ by assumption, [27, Lemma 2.12] shows that $C(\underline{E})$ is polynomially factorizable if and only if for every $g \in \mathbb{Z}[X]$, $\mathbb{P}_{g, C(\underline{E})}$ is finite. Hence, (3) and (4) are equivalent. $\square$

Let Λ and R be as in Theorem 4.34. It is helpful to note that $\Lambda = \emptyset$ if and only if $R = \mathbb{Q}[X]$, and, if Λ is finite, then R is always a Krull domain. The next theorem is the global version of Theorem 4.15 and describes the class of polynomial Krull Domains. By a result of Heinzer, this is precisely the class of integrally closed Noetherian domains between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ (see [18]).

Theorem 4.35. *Let R be a polynomial Krull domain. Then*

$$(4.36) \quad R = \bigcap_{p \in \Lambda} \bigcap_{j=1}^{m_p} \mathbb{Z}_{(p), \alpha_{p,j}, \delta_{p,j}} \cap \mathbb{Q}[X] = \text{Int}_{\mathbb{Q}}(\underline{E}, \mathbb{O})$$

for some subset $\Lambda \subseteq \mathbb{P}$, where $(\alpha_{p,j}, \delta_{p,j}) \in \mathbb{O}_p^{\text{br}} \times (\mathbb{Q}_{\geq 0} \cup \{\infty\})$ with $\alpha_{p,j}$ transcendental over $\mathbb{Q}$ if $\delta_{p,j} = \infty$ for each $p \in \Lambda$ and $j = 1, \dots, m_p$; $\underline{E} = \prod_{p \in \Lambda} E_p$ with $E_p = \bigcup_{j=1}^{m_p} \bar{B}_p(\alpha_{p,j}, \delta_{p,j})$; and $C(\underline{E}) = \prod_{p \in \Lambda} \{\alpha_{p,1}, \dots, \alpha_{p,m_p} \mid \forall j = 1, \dots, m_p, \delta_{p,j} > 0\}$ is polynomially factorizable.

Moreover, if $ht(M_{(p), \alpha_{p,j}, \delta_{p,j}} \cap R) = 1$ for all $p \in \Lambda$ and $j = 1, \dots, m_p$, then

$$Cl(R) = \bigoplus_{p \in \Lambda} (\mathbb{Z}/d_p \mathbb{Z} \oplus \mathbb{Z}^{m_p-1}),$$

where $e_{p,j} = e(\mathbb{Z}_{(p), \alpha_{p,j}, \delta_{p,j}} | \mathbb{Z}_{(p)})$ for $j = 1, \dots, m_p$ and $d_p = \gcd(e_{p,1}, \dots, e_{p,m_p})$ for all $p \in \Lambda$.

Proof. As we remarked in (4.1), we have

$$R = \bigcap_{p \in \Lambda} \left(\bigcap_{j=1}^{m_p} W_{p,j} \right) \cap \mathbb{Q}[X],$$

for some subset $\Lambda \subseteq \mathbb{P}$ where $W_{p,j}$ is a DVR of $\mathbb{Q}(X)$ lying over $\mathbb{Z}_{(p)}$ for each $j = 1, \dots, m_p$ and $p \in \mathbb{P}$.

By Theorem 3.6, for each $p \in \Lambda$, there exist $(\alpha_{p,j}, \delta_{p,j}) \in \mathbb{O}_p^{\text{br}} \times (\mathbb{Q}_{\geq 0} \cup \{\infty\})$ such that $W_{p,j} = \mathbb{Z}_{(p), \alpha_{p,j}, \delta_{p,j}}$, for each $j = 1, \dots, m_p$. In particular, R has the stated form.

Finally, $C(\underline{E}) = \prod_{p \in \Lambda} \{\alpha_{p,1}, \dots, \alpha_{p,m_p} \mid \forall j = 1, \dots, m_p, \delta_{p,j} > 0\}$ is polynomially factorizable by Theorem 4.34 and $\text{Cl}(R) = \bigoplus_{p \in \Lambda} (\mathbb{Z}/d_p \mathbb{Z} \oplus \mathbb{Z}^{m_p-1})$ by the proof of [10, Theorem 2.4(4)]. $\square$

In the next corollary, we completely describe UFDs between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$.

Corollary 4.37. *Let R be a UFD between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$. Then there exist a subset $\Lambda \subseteq \mathbb{P}$ and pairs $(\alpha_p, \delta_p) \in \mathbb{O}_p^{br} \times (\mathbb{Z}_{\geq 0} \cup \{\infty\})$ such that*

$$(4.38) \quad R = \bigcap_{p \in \Lambda} \mathbb{Z}_{(p), \alpha_p, \delta_p} \cap \mathbb{Q}[X] = \bigcap_{p \in \Lambda} \text{Int}_{\mathbb{Q}}(\bar{B}_p(\alpha_p, \delta_p), \mathbb{O}_p)$$

where, for each $p \in \Lambda$, α_p is unramified over $\mathbb{Q}_p$ and is transcendental over $\mathbb{Q}$ if $\delta_p = \infty$ and the set $C(\underline{E}) = \prod_{p \in \Lambda} \{\alpha_p \mid \delta_p > 0\}$ is polynomially factorizable.

Proof. It is well-known that a UFD is a Krull domain with trivial class group (see for example [13, Proposition 6.1]). By Theorem 4.35, R is as in (4.36) (and so, in particular, it is a Noetherian domain by [18]). By the characterization of the class group of R in the aforementioned Theorem and also Theorem 4.24, R is as in the statement of the corollary. Note that the second equality of (4.38) follows from Remark 3.12. $\square$

Corollary 4.39. *Let $\Lambda \subseteq \mathbb{P}$ be a subset and $(\alpha_{p,j}, \delta_{p,j}), (\alpha_{p,j}, \delta'_{p,j}) \in \mathbb{O}_p^{br} \times (\mathbb{Z}_{\geq 0} \cup \{\infty\})$ with $\delta_{p,j} \leq \delta'_{p,j}$, and $\alpha_{p,j}$ transcendental over $\mathbb{Q}$ if $\delta_{p,j} = \infty$, for each $p \in \Lambda$ and $j = 1, \dots, m_p$. Let*

$$R = \bigcap_{p \in \Lambda} \bigcap_{j=1}^{m_p} \mathbb{Z}_{(p), \alpha_{p,j}, \delta_{p,j}} \cap \mathbb{Q}[X] \text{ and } R' = \bigcap_{p \in \Lambda} \bigcap_{j=1}^{m_p} \mathbb{Z}_{(p), \alpha_{p,j}, \delta'_{p,j}} \cap \mathbb{Q}[X].$$

Let $\Delta = \{p \in \Lambda \mid 0 = \delta_{p,j} < \delta'_{p,j} \text{ for some } j = 1, \dots, m_p\}$ and assume that Δ is a finite set. Then the following statements hold.

- (1) R is a Krull domain if and only if R' is a Krull domain.
- (2) Let R be a Krull domain. Then $\text{Cl}(R')$ is a factor group of $\text{Cl}(R)$. Moreover, if $\delta'_{p,j} < \infty$ for all $\delta_{p,j}$ with $\delta_{p,j} < \infty$, then $\text{Cl}(R') = \text{Cl}(R)$.
- (3) If R is a UFD, then R' is also a UFD.

Proof. (1) For each $g \in \mathbb{Z}[X]$, let

$$\begin{aligned} \mathbb{D}_{g, \underline{E}}(R) &= \{p \in \Lambda \mid \exists j \in \{1, \dots, m_p\} \text{ such that } g \in \mathbb{Z}[X] \cap M_{(p), \alpha_{p,j}, \delta_{p,j}}\} \\ \mathbb{D}_{g, \underline{E}}(R') &= \{p \in \Lambda \mid \exists j \in \{1, \dots, m_p\} \text{ such that } g \in \mathbb{Z}[X] \cap M_{(p), \alpha_{p,j}, \delta'_{p,j}}\}. \end{aligned}$$

Then, by Theorem 4.34, it suffices to show that $\mathbb{D}_{g, \underline{E}}(R)$ is finite if and only if $\mathbb{D}_{g, \underline{E}}(R')$ is finite.

It is clear that $\mathbb{Z}[X] \cap M_{(p), \alpha_{p,j}, \delta_{p,j}} \subseteq \mathbb{Z}[X] \cap M_{(p), \alpha_{p,j}, \delta'_{p,j}}$ for each $p \in \Lambda$ and $j = 1, \dots, m_p$. Hence, $\mathbb{D}_{g, \underline{E}}(R) \subseteq \mathbb{D}_{g, \underline{E}}(R')$. Moreover, if $\delta_{p,j} \neq 0$

for $j = 1, \dots, m_p$, then $\mathbb{Z}[X] \cap M_{(p), \alpha_{p,j}, \delta_{p,j}} = \mathbb{Z}[X] \cap M_{(p), \alpha_{p,j}, \delta'_{p,j}}$ by Proposition 4.30. Hence, $\mathbb{D}_{g,E}(R') \setminus \mathbb{D}_{g,E}(R) \subseteq \Delta$, and since Δ is finite, $\mathbb{D}_{g,E}(R') \setminus \mathbb{D}_{g,E}(R)$ is finite. Thus $\mathbb{D}_{g,E}(R)$ is finite if and only if $\mathbb{D}_{g,E}(R')$ is also finite.

(2) Let the notation be as in Theorems 4.10 and 4.35. We may assume that the representation of R above is irredundant. Hence, by Proposition 3.16, the representation of R' above is also irredundant. Note that $(\alpha_{p,j}, \delta_{p,j})$ is a minimal pair if $\delta_{p,j} \in \mathbb{Q}$, so if $\delta'_{p,j} < \infty$, then $(\alpha_{p,j}, \delta'_{p,j})$ is a minimal pair because $\bar{B}_p(\alpha_{p,j}, \delta'_{p,j}) \subseteq \bar{B}_p(\alpha_{p,j}, \delta_{p,j})$. Hence $\text{Cl}(R') = \bigoplus_{p \in \Lambda} (\mathbb{Z}/d'_p \mathbb{Z} \oplus \mathbb{Z}^{m_p-1})$ for

some positive integer d'_p with $d'_p | d_p$ for each $p \in \Lambda$ by Theorem 4.10. Hence, $\mathbb{Z}/d'_p \mathbb{Z} \cong (\mathbb{Z}/d_p \mathbb{Z}) / (d'_p \mathbb{Z}/d_p \mathbb{Z})$, so if we let $H = \bigoplus_{p \in \Lambda} (d'_p \mathbb{Z}/d_p \mathbb{Z} \oplus \{0\}^{m_p-1})$,

then H is a subgroup of $\text{Cl}(D)$ and $\text{Cl}(R)/H = \text{Cl}(R')$. In particular, if $\delta'_{p,j} < \infty$ for all $\delta_{p,j}$ with $\delta_{p,j} < \infty$, then $d_p = d'_p$ by Theorem 4.10 again. Thus $\text{Cl}(R) = \text{Cl}(R')$.

(3) This follows by (2) above because a UFD is a Krull domain with trivial class group. $\square$

Example 4.40. Let $R = \bigcap_{p \in \mathbb{P}} \mathbb{Z}_{(p), 0, 0} \cap \mathbb{Q}[X]$. Then $R = \mathbb{Z}[X]$, and hence R is a Krull domain (in fact, a UFD). However, if $R' = \bigcap_{p \in \mathbb{P}} \mathbb{Z}_{(p), 0, p} \cap \mathbb{Q}[X]$, then R is not a Krull domain because $X \in X\mathbb{Z}_{(p), 0, p} = p\mathbb{Z}_{(p), 0, p}$ for all $p \in \mathbb{P}$. Therefore the condition of Δ being finite is necessary for Corollary 4.39.

4.3.4. Construction of Krull domains with prescribed class group. For convenience, we say that a polynomial Krull domain R is a *pure* Krull domain if Q is not maximal for all $Q \in X^1(R)$ with $Q \cap \mathbb{Z} \neq (0)$. By Proposition 4.28, if R is a polynomial Krull domain that is not Dedekind, then R can be represented as an intersection of a Dedekind domain R_1 and a pure Krull domain R_2 , i.e., $R = R_1 \cap R_2$. The divisor class group of a polynomial Krull domain is a direct sum of a countable family of finitely generated abelian groups by Theorem 4.35. Conversely, if G is a direct sum of a countable family of finitely generated abelian groups, then there is a Dedekind domain between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ with ideal class group G [27, Theorem 3.1].

We next construct a pure Krull domain with prescribed divisor class group.

Proposition 4.41. *Let G be a finitely generated abelian group. Then there exists a pure Krull domain R between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ such that $\text{Cl}(R) = G$.*

Proof. Suppose $G = \mathbb{Z}/e_1 \mathbb{Z} \oplus \dots \oplus \mathbb{Z}/e_n \mathbb{Z} \oplus \mathbb{Z}^m$ for some integers $e_1, \dots, e_n \in \mathbb{N}$ and $m \geq 0$. Let $p_1, \dots, p_n \in \mathbb{P}$ be distinct.

We pick $m+1$ distinct elements $\alpha_{p_n, 1}, \dots, \alpha_{p_n, m+1}$ in $\mathbb{Z}_{p_n}$ which are transcendental over $\mathbb{Q}$. Let $N = \max\{v_{p_n}(\alpha_{p_n, i} - \alpha_{p_n, j}) \mid i \neq j \in \{1, \dots, m+1\}\}$. Let $W_{p_n, j} = \mathbb{Z}_{(p_n), \alpha_{p_n, j}, N+1/e_n}$, for $j \in \{1, \dots, m+1\}$. Note that the $W_{p_n, j}$'s are distinct (because $v_{p_n}(\alpha_{p_n, i} - \alpha_{p_n, j}) < N + 1/e_n$ for each $i \neq j$) and

$(\alpha_{p_n,j}, N + 1/e_n)$ is a minimal pair for each $j \in \{1, \dots, m+1\}$. Then by Theorem 4.10, (4) and Proposition 4.22 we have

$$e_{p_n,j} = e(W_{p_n,j} \mid \mathbb{Z}_{(p_n)}) = e(N + 1/e_n \mid \mathbb{Q}_p) = e_n.$$

By the same argument, we can also choose α_{p_i} in $\mathbb{Z}_{p_i}$ which is transcendental over $\mathbb{Q}$, so that $(\alpha_{p_i}, 1/e_i)$ is minimal pair for $i = 1, \dots, n-1$. Let $W_{p_i} = \mathbb{Z}_{(p_i), \alpha_{p_i}, 1/e_i}$ for $i = 1, \dots, n-1$ so that, if we set $R = (\bigcap_{i=1}^{n-1} W_{p_i}) \cap (\bigcap_{j=1}^{m+1} W_{p_n,j} \cap \mathbb{Q}[X])$, we have that R is a Krull domain such that $\text{Cl}(R) = G$ by Theorem 4.35 and Q is not maximal for all $Q \in X^1(R)$ with $Q \cap \mathbb{Z} \neq (0)$ by Theorem 4.10(3). $\square$

Theorem 4.42. *Let G be a direct sum of a countable family $\{G_i \mid i \in I\}$ of finitely generated abelian groups (which are not necessarily distinct). Then there exists a pure Krull domain R between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ with $\text{Cl}(R) = G$. Moreover, for each $i \in I$, there is a multiplicative subset S_i of $\mathbb{Z}$ such that R_{S_i} is a pure Krull domain with $\text{Cl}(R_{S_i}) = G_i$.*

Proof. For each $i \in I$, let $G_i = \mathbb{Z}/e_{i1}\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/e_{in_i}\mathbb{Z} \oplus \mathbb{Z}^{m_i}$ for some $e_{i1}, \dots, e_{in_i} \geq 1$, $n_i \geq 1$ and $m_i \geq 0$. Without loss of generality, we may suppose that we have a partition of $\mathbb{P}$ into a family of finite subsets which is indexed by I , namely, $\{\mathbb{P}_i\}_{i \in I}$ is a partition of $\mathbb{P}$ with $|\mathbb{P}_i| = n_i$ for each $i \in I$. By Proposition 4.41, there exists a pure Krull domain R_i , $\mathbb{Z}[X] \subseteq R_i \subset \mathbb{Q}[X]$, such that $\text{Cl}(R_i) = G_i$.

Now set $R = \bigcap_{i \in I} R_i$. Note that if $S_i = \mathbb{Z} \setminus \bigcup_{p \in \mathbb{P}_i} p\mathbb{Z}$ for each $i \in I$, then $R_{S_i} = R_i$ by Lemma 4.3, thus R_{S_i} is a pure Krull domain with $\text{Cl}(R_{S_i}) = G_i$. However, it may be the case that R is not a Krull domain, that is, it has not the finite character property.

Let $i \in I$ be such that G_i is not trivial, and let p be a fixed prime in the finite set $\mathbb{P}_i$. Suppose that for the construction of R_i we chose in $\mathbb{Z}_p$ $k_p \geq 1$ distinct elements, say for example $E_p = \{\alpha_{p,1}, \dots, \alpha_{p,k_p}\}$; let also $\delta_{p,j}$ be the radius of the ultrametric ball of center $\alpha_{p,j}$, for $j = 1, \dots, k_p$. We then replace the elements in E_p by possibly other k_p distinct α_p 's in $\mathbb{Z}_p$ which lie in the same residue class $[\log(p)] + p\mathbb{Z}_p$, such that these α_p 's are transcendental over $\mathbb{Q}$. We also modify the corresponding $\delta_{p,j}$'s accordingly so that the previous facts about the corresponding ring R_i still hold. If G_i is trivial, then we simply choose an element $\alpha_p \in \mathbb{Z}_p$. We claim that the set

$$C(\underline{E}) = \prod_{p \in \mathbb{P}} \{\alpha_{p,1}, \dots, \alpha_{p,k_p} \in \mathbb{Z}_p \mid \delta_{p,j} > 0\}$$

is polynomially factorizable, so by Theorems 4.34 and 4.10(3), R is a pure Krull domain. Let $g \in \mathbb{Z}[X]$ and let $\underline{\alpha} = (\alpha_p) \in C(\underline{E})$. Then we have

$$g(\alpha_p) \equiv g([\log(p)]) \pmod{p}$$

Since for p sufficiently large, $g([\log(p)])$ is not divisible by p we get that $\mathbb{P}_{g,C(\underline{E})}$ is finite. $\square$

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