

Extending valuations of local domains to complete local domains without changing the value group.

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ABSTRACT

Let (R, m, k) be an excellent local noetherian domain with field of fractions K . Let

$$\nu : K^* \rightarrow \Gamma$$

be a valuation centered at R and let R_ν be the corresponding valuation ring of K , dominating R . Denote by $\widehat{R}$ the m -adic completion of R . In the applications of valuation theory to commutative algebra and the study of singularities, one is often induced to replace R by its m -adic completion $\widehat{R}$ and ν by a suitable extension $\widehat{\nu}_-$ to $\frac{\widehat{R}}{P}$ for a suitably chosen prime ideal P , such that

$$P \cap R = (0).$$

In a previous article we gave a systematic description of all such extensions $\widehat{\nu}_-$ and defined the notion of tight extensions that are of particular interest for applications (see Herrera, Olalla, Spivakovsky and Teissier, *Extending a valuation centered in a local domain to its formal completion*, Proc. London Math. Soc. (3) 105 (2012) 571–621). If $\widehat{\nu}_-$ is a tight extension then its graded algebra is birational to that of ν (the converse is not known and might not be true). In particular, the value group of $\widehat{\nu}_-$ is Γ . The existence of tight extensions was conjectured by the last author (see Teissier, *Valuations, deformations, and toric geometry*, Fields Institute Communications, 33, 2003, 361–459). In the present paper we give a proof of Teissier’s conjecture. An intended application of this result is an important step in two recent approaches to local uniformization in positive characteristic.

1. Introduction

All the rings in this paper will be commutative with 1.

Let (R, m, k) be a local noetherian domain with field of fractions K and R_ν a valuation ring, birationally dominating R . Let $\nu : K^* \rightarrow \Gamma$ be the corresponding valuation of K , centered at R . Here Γ is the value group; it is totally ordered and abelian. Let $\widehat{R}$ denote the m -adic completion of R . In the applications of valuation theory to commutative algebra and the study of singularities, one is often induced to replace R by its m -adic completion $\widehat{R}$ and ν by a suitable extension $\widehat{\nu}_-$ to $\frac{\widehat{R}}{P}$ for a suitably chosen prime ideal P , such that $P \cap R = (0)$. In the paper [8] we gave, assuming that R is excellent, a systematic description of all such extensions $\widehat{\nu}_-$ and among them identified the class of tight extensions. The existence of tight extensions (see Conjecture 1 of [8]) is the restatement in the framework of [8] of a conjecture of the last author going back to 2003 (see [15], *proposition* 5.19).

The main result of the present paper is a proof of Teissier’s conjecture:

Given an excellent local domain R and a valuation ν of R that is strictly positive on its maximal ideal m , there exists a prime ideal P of the m -adic completion $\widehat{R}$

such that $P \cap R = (0)$ and an extension $\hat{\nu}_-$ of ν to $\frac{\hat{R}}{P}$ such that the natural inclusion

$$\mathrm{gr}_\nu R \subset \mathrm{gr}_{\hat{\nu}_-} \hat{R}/P \quad (1.1)$$

of the associated graded rings is scalewise-birational (see the definitions below). In particular, the valuations ν and $\hat{\nu}_-$ have the same value group.

The only assumption about R we ever used in [8] is a weaker and more natural condition than excellence: namely, we require the completion homomorphism $R \rightarrow \hat{R}$ to be regular. Local rings R having this property are called G-rings or quasi-excellent.

One specific application we have in mind has to do with the approaches to proving the Local Uniformization Theorem in arbitrary characteristic which are described in [14] and [15].

These approaches to local uniformization both make important use, in different ways, of the structure of the graded ring $\mathrm{gr}_\nu R$ associated to the filtration of an excellent local domain R defined by the valuation ν . Extending ν to a valuation $\hat{\nu}_-$ of a complete local domain $\hat{R}/P$ in such a manner that the corresponding extension $\mathrm{gr}_\nu R \subset \mathrm{gr}_{\hat{\nu}_-} \hat{R}/P$ of graded rings is scalewise-birational allows one to have access to the many advantages of completeness without losing the algebraicity of the centers of blowing-up (in [14]) or of the embeddings into appropriate larger affine spaces where a birational toric map will provide local uniformization (in [15]).

Let r denote the (real) rank of ν . Let $(0) = \Delta_r \subsetneq \Delta_{r-1} \subsetneq \cdots \subsetneq \Delta_0 = \Gamma$ be the isolated subgroups of Γ and $P_0 = (0) \subsetneq P_1 \subseteq \cdots \subseteq P_r = m$ the prime valuation ideals of R , which need not, in general, be distinct. Assuming that R is excellent, in [8] we canonically associated to ν a chain of $2r + 2$ prime ideals

$$H_0 \subset H_1 \subset \cdots \subset H_{2r} = H_{2r+1} = m\hat{R},$$

satisfying $H_{2\ell} \cap R = H_{2\ell+1} \cap R = P_\ell$ and such that $H_{2\ell}$ is a minimal prime of $P_\ell \hat{R}$ for $0 \leq \ell \leq r$. We called H_i the *i-th implicit prime ideal* of $\hat{R}$, associated to R and ν . The role of these ideals is explained at the beginning of section 2.1 below.

The ideals H_i behave well under local blowing ups along ν (that is, birational local homomorphisms $R \rightarrow R'$ such that ν is centered in R'). This means that given a local blowing up $R \rightarrow R'$ along ν , the *i*-th implicit prime ideal H'_i of $\hat{R}'$ has the property that $H'_i \cap \hat{R} = H_i$. In this situation we say that the collection $\{H'_i\}$ forms a **tree** (see subsection 1.1).

For a prime ideal P in a ring R , $\kappa(P)$ will denote the residue field $\frac{R_P}{P R_P}$.

Let $(0) = \mathbf{m}_0 \subsetneq \mathbf{m}_1 \subsetneq \cdots \subsetneq \mathbf{m}_{r-1} \subsetneq \mathbf{m}_r = \mathbf{m}_\nu$ be the prime ideals of the valuation ring R_ν . By definitions, our valuation ν is a composition of r rank one valuations, $\nu = \nu_1 \circ \nu_2 \cdots \circ \nu_r$, where ν_ℓ is a valuation of the field $\kappa(\mathbf{m}_{\ell-1})$, centered at $\frac{(R_\nu)_{\mathbf{m}_\ell}}{\mathbf{m}_{\ell-1}(R_\nu)_{\mathbf{m}_\ell}}$. In particular, ν_ℓ defines a valuation on $\frac{R}{P_{\ell-1}}$ with value group $\frac{\Delta_{\ell-1}}{\Delta_\ell}$ by restriction (see [20], Chapter VI, §10, p. 43 for the definition of composition of valuations; more information is given below in subsection 1.1, where we interpret each $\mathbf{m}_\ell$ as the limit of a tree of ideals). For each integer $\ell \in \{1, \dots, r\}$, let $\mu_\ell := \nu_\ell \circ \nu_{\ell+1} \circ \cdots \circ \nu_r$; it is the residual valuation induced by ν on $\frac{R}{P_{\ell-1}}$.

1.1. Trees.

We consider birational ν -extensions $R \rightarrow R'$ of local rings, that is, injective homomorphisms such that R' is an R -algebra essentially of finite type with field of fractions K , dominated by R_ν (in particular, we have $m' \cap R = m$). Such extensions form a direct system $\{R'\}$ with

$$\lim_{\substack{\longrightarrow \\ R'}} R' = R_\nu. \quad (1.2)$$

We will consider many direct systems of rings and of ideals indexed by $\{R'\}$; direct limits will always be taken with respect to the direct system $\{R'\}$.

DEFINITION 1. A **tree** of R' -algebras is a direct system $\{S'\}$ of rings, indexed by the directed set $\{R'\}$, where S' is an R' -algebra. Note that the tree maps are not required to be injective or birational. A morphism $\{S'\} \rightarrow \{T'\}$ of trees is the datum of a map of R' -algebras $S' \rightarrow T'$ for each R' commuting with the tree morphisms $S' \rightarrow S''$ and $T' \rightarrow T''$ for each map $R' \rightarrow R''$.

DEFINITION 2. Let $\{S'\}$ be a tree of R' -algebras. For each S' , let I' be an ideal of S' . We say that $\{I'\}$ is a **tree of ideals** if for every arrow $b_{S'S''}: S' \rightarrow S''$ in our direct system, we have $b_{S'S''}^{-1}I'' = I'$. We have the obvious notion of inclusion of trees of ideals. In particular, we may speak about chains of trees of ideals.

EXAMPLES. For each non-negative element $\beta \in \Gamma$, the valuation ideals $\mathcal{P}'_\beta \subset R'$ of value β form a tree of ideals of $\{R'\}$. Similarly, the i -th prime valuation ideals $P'_i \subset R'$ form a tree. If $r \cdot k \nu = r$, the prime valuation ideals P'_i give rise to a chain

$$(0) = P'_0 \subsetneq P'_1 \subseteq \cdots \subseteq P'_r = m' \quad (1.3)$$

of trees of prime ideals of $\{R'\}$.

In [8] we proved that the implicit prime ideals H'_i form a tree of ideals of $\widehat{R}$.

We showed that specifying an extension $\widehat{\nu}_-$ of ν as above is equivalent to specifying a chain

$$\tilde{H}'_0 \subset \tilde{H}'_1 \subset \cdots \subset \tilde{H}'_{2r} = m' \widehat{R}' \quad (1.4)$$

of trees of prime valuation ideals of $\widehat{R}'$ such that $H'_\ell \subset \tilde{H}'_\ell$ for all $\ell \in \{0, \dots, 2r\}$, and valuations $\widehat{\nu}_1, \widehat{\nu}_2, \dots, \widehat{\nu}_{2r}$, where $\widehat{\nu}_i$ is a valuation of the field $k_{\widehat{\nu}_{i-1}}$ (the residue field of the valuation ring $R_{\widehat{\nu}_{i-1}}$), arbitrary when i is odd and satisfying certain conditions, coming from the valuation $\nu_{\frac{i}{2}}$, when i is even.

Recall that for each $\beta \in \Gamma$ one associates to β the valuation ideals

$$\begin{aligned} \mathcal{P}_\beta(R) &= \{x \in R / \nu(x) \geq \beta\} \\ \mathcal{P}_\beta^+(R) &= \{x \in R / \nu(x) > \beta\} \end{aligned}$$

of R (with the convention that $\nu(0) = \infty$, an element larger than any element of Γ) and the graded ring

$$\mathrm{gr}_\nu R = \bigoplus_{\beta \in \Gamma_+} \frac{\mathcal{P}_\beta(R)}{\mathcal{P}_\beta^+(R)}.$$

Note that if the ring R is noetherian, the semigroup $\Phi \subset \Gamma_{\geq 0}$ of values of ν on $R \setminus \{0\}$ is well ordered. If $R \rightarrow R'$ is a birational extension of local domains dominated by R_ν , we will write $\mathcal{P}'_\beta$ for $\mathcal{P}_\beta(R')$.

NOTATION. Let $\mathcal{T} = \{R'\}$, the tree defined above. For a ring $R' \in \mathcal{T}$, we shall denote by $\mathcal{T}(R')$ the subtree of $\mathcal{T}$ consisting of all the ν -extensions R'' of R' .

Let

$$\Gamma \hookrightarrow \widehat{\Gamma} \quad (1.5)$$

be an extension of ordered groups of the same rank. Let

$$(0) = \widehat{\Delta}_r \subsetneq \widehat{\Delta}_{r-1} \subsetneq \cdots \subsetneq \widehat{\Delta}_0 = \widehat{\Gamma}$$

be the isolated subgroups of $\widehat{\Gamma}$, so that the inclusion (1.5) induces inclusions

$$\Delta_\ell \hookrightarrow \widehat{\Delta}_\ell \quad \text{and} \quad (1.6)$$

$$\frac{\Delta_\ell}{\Delta_{\ell+1}} \hookrightarrow \frac{\widehat{\Delta}_\ell}{\widehat{\Delta}_{\ell+1}}. \quad (1.7)$$

Let $G \hookrightarrow \widehat{G}$ be an extension of graded algebras without zero divisors, such that G is graded by Γ_+ and $\widehat{G}$ by $\widehat{\Gamma}_+$. The graded algebra G is endowed with a natural valuation with value group Γ and similarly for $\widehat{G}$ and $\widehat{\Gamma}$. Both of these natural valuations will be denoted by *ord*.

DEFINITION 3. We say that the extension $G \hookrightarrow \widehat{G}$ is **scalewise-birational** if $G_0 = \widehat{G}_0$ and for all homogeneous elements $x \in \widehat{G}$ and $\ell \in \{1, \dots, r-1\}$ such that $\text{ord } x \in \widehat{\Delta}_\ell$ there exists $y \in G$ such that $\text{ord } y \in \Delta_\ell$ and $xy \in G$.

Consider a tree of prime ideals H' of $\widehat{R}'$ with $H' \cap R' = (0)$ and a valuation $\widehat{\nu}_-$, centered at $\lim_{\rightarrow} \frac{\widehat{R}'}{\widehat{H}'}$.

DEFINITION 4. We say that $\widehat{\nu}_-$ is a **scalewise-birational extension** of ν if for each R' the inclusion

$$\text{gr}_\nu R' \subset \text{gr}_{\widehat{\nu}_-} \widehat{R}' / H' \quad (1.8)$$

is scalewise-birational.

Of course, if $\widehat{\nu}_-$ is a scalewise-birational extension of ν then the inclusion (1.8) is birational in the usual sense (that is, induces equality of fraction fields) which, in turn, implies that $\widehat{\Gamma} = \Gamma$.

The main result of the present paper is:

THEOREM 1.1. *There exists a tree of prime ideals $\tilde{H}'_0$ of $\widehat{R}'$ with $\tilde{H}'_0 \cap R' = (0)$ and a scalewise-birational extension $\widehat{\nu}_-$, centered at $\lim_{\rightarrow} \frac{\widehat{R}'}{\widehat{H}'_0}$.*

The example given in Remark 5.20, 4) of [15] shows that the morphism of associated graded rings is not an isomorphism in general.

Acknowledgement. We acknowledge earlier work in this area by W. Heinzer and J. Sally [6] (which deals with the case $\dim R = 2$) as well as by the school of S. D. Cutkosky [2, 3] that helped inspire some of the authors to think about the subject.

2. A review of some of the main definitions and results from [7]

2.1. Definition and first properties of implicit prime ideals

Let $0 \leq \ell \leq r$. We recall the definition of one of the main objects in the theory, the *implicit prime ideals* of ν in $\widehat{R}$. Broadly speaking, the odd-numbered implicit prime ideals represent the fact that elements of $\widehat{R}$ can be limits of sequences of elements of R of ever increasing valuation, tending to infinity at least for one of the valuations with which ν is composed, while even-numbered implicit ideals represent the fact that while some quotients $\frac{R}{P_\ell}$ may not be analytically irreducible, the valuation ν “chooses” one of the irreducible components of the

completion. These representations must then be “stabilized” with respect to the tree $\mathcal{T}$ in the way shown by equation (2.1) below.

The $(2\ell + 1)$ -st **implicit prime ideal** $H_{2\ell+1} \subset \widehat{R}$ is defined by

$$H_{2\ell+1} = \bigcap_{\beta \in \Delta_\ell} \left(\left(\lim_{\overrightarrow{R'}} \mathcal{P}'_\beta \widehat{R}' \right) \cap \widehat{R} \right), \quad (2.1)$$

where R' ranges over $\mathcal{T}$. As usual, we think of (2.1) as a tree equation: if we replace R by any other $R'' \in \mathcal{T}$ in (2.1), it defines the corresponding ideal $H''_{2\ell+1} \subset \widehat{R''}$.

PROPOSITION 2.1. ([8], Proposition 3.1) *We have $H'_{2\ell+1} \cap R' = P'_\ell$.*

The implicit ideal $H_{2\ell}$ is defined as the unique minimal prime ideal of $P_\ell \widehat{R}$ contained in $H_{2\ell+1}$.

Here the uniqueness follows from the fact that R being excellent (or a G -ring), the ring $\widehat{R} \otimes_R \kappa(P_\ell)$ is regular, hence a domain. Thus the only minimal prime of $\widehat{R} \otimes_R \kappa(P_\ell)$ is (0) and hence the unique minimal prime of $P_\ell \widehat{R}$ in $\widehat{R}$ is the kernel of the natural map $\widehat{R} \rightarrow \widehat{R} \otimes_R \kappa(P_\ell)$.

In ([8], Theorem 8.1) we prove the primality of the implicit ideals.

2.2. A classification of extensions of ν to $\widehat{R}$.

One is naturally led to consider the more general problem of extending ν not only to rings of the form $\frac{\widehat{R}}{P}$ but also to the ring $\lim_{\overrightarrow{R'}} \frac{\widehat{R}'}{P'}$, where P' is a tree of prime ideals of $\widehat{R'}$, such that $P' \cap R' = (0)$.

In §5 of [8], we gave a systematic description of all the possible extensions $\widehat{\nu}_-$ of ν to $\lim_{\overrightarrow{R'}} \frac{\widehat{R}'}{P'}$, as compositions of $2r$ valuations,

$$\widehat{\nu}_- = \widehat{\nu}_1 \circ \cdots \circ \widehat{\nu}_{2r}, \quad (2.2)$$

satisfying certain conditions. We associate to each extension $\widehat{\nu}_-$ of ν a chain

$$\tilde{H}'_0 \subset \tilde{H}'_1 \subset \cdots \subset \tilde{H}'_{2r} = m' \widehat{R}' \quad (2.3)$$

of prime $\widehat{\nu}_-$ -ideals, corresponding to the decomposition (2.2), satisfying

$$\tilde{H}'_{2\ell} \cap R' = \tilde{H}'_{2\ell+1} \cap R' = P'_\ell. \quad (2.4)$$

By definitions, for $1 \leq i \leq 2r$, $\widehat{\nu}_i$ is a valuation of the field $k_{\widehat{\nu}_{i-1}}$.

PROPOSITION 2.2. ([8], Proposition 5.3) *We have*

$$H'_i \subset \tilde{H}'_i \quad \text{for all } i \in \{0, \dots, 2r\}. \quad (2.5)$$

One of the main theorems of [8] (Theorem 5.4) says that specifying the valuation $\widehat{\nu}_-$ is equivalent to specifying the following data:

- (1) A chain of trees (2.3) of prime ideals of $\widehat{R'}$ satisfying (2.5) and some additional conditions.
- (2) For each i , $1 \leq i \leq 2r$, a valuation $\widehat{\nu}_i$ of $k_{\widehat{\nu}_{i-1}}$ (where $\widehat{\nu}_0$ is taken to be the trivial valuation by convention), whose restriction to $\lim_{\overrightarrow{R'}} \kappa(\tilde{H}'_{i-1})$ is centered at the local ring $\lim_{\overrightarrow{R'}} \frac{\widehat{R}'_{\tilde{H}'_{i-1}}}{\tilde{H}'_{i-1} \widehat{R}'_{\tilde{H}'_{i-1}}}$.

The data $\{\widehat{\nu}_i\}_{1 \leq i \leq 2r}$ is subject to the following additional condition: if $i = 2\ell$ is even then $rk \widehat{\nu}_i = 1$ and $\widehat{\nu}_i$ is an extension of ν_i to $k_{\widehat{\nu}_{i-1}}$ (which is naturally an extension of $k_{\nu_{i-1}}$).

In particular, such extensions $\widehat{\nu}_-$ always exist, and usually there are plenty of them.

2.3. Tight extensions and scalewise birationality

In §6 of [8] we define and study, among other things, a class of extensions $\hat{\nu}_-$ which are particularly useful for the applications called the **tight** extensions (see [8, Definition 6.1]). One of the properties of tight extensions is the fact that

$$\tilde{H}'_{2\ell} = \tilde{H}'_{2\ell+1} \quad \text{for } 1 \leq \ell \leq r. \quad (2.6)$$

PROPOSITION 2.3. ([8], Proposition 6.7) *The extension $\hat{\nu}_-$ is tight if and only if for each R' in our direct system the natural graded algebra extension $gr_\nu R' \rightarrow gr_{\hat{\nu}_-} \widehat{R'}$ is scalewise-birational.*

REMARK 1. Proposition 2.3 allows us to rephrase Conjecture 1 of [8] as follows: the valuation ν admits at least one tight extension $\hat{\nu}_-$.

The only material from [8] used in the proof of the main theorem of the present paper is the definition (2.1) of implicit ideals, their primality (Theorem 8.1) and the equality

$$H'_{2\ell+1} \cap R' = P'_\ell$$

(Proposition 3.1), but not the somewhat technical Theorem 5.4.

3. A proof of the existence of a scalewise-birational extension $\hat{\nu}_-$

The purpose of this section is to prove Theorem 1.1 by constructing a valuation $\hat{\nu}_-$ whose associated graded algebra is a scalewise-birational extension of $gr_\nu R$.

The construction consists of describing the trees of ideals $\tilde{H}'_i$, $0 \leq i \leq 2r$ and, for each i , a valuations $\hat{\nu}_i$ of the residue field $k_{\hat{\nu}_{i-1}}$, such that $\hat{\nu}_- = \hat{\nu}_1 \circ \dots \circ \hat{\nu}_{2r}$. More precisely, for $\ell \in \{1, \dots, r\}$, we will construct, recursively in the descending order of ℓ , a tree $\tilde{H}'_{2\ell-1}$ of prime ideals of $\widehat{R'}$, $R' \in \mathcal{T}$, such that $\tilde{H}'_{2\ell-1} \cap R' = P'_{\ell-1}$, and a scalewise-birational extension $\hat{\mu}_{2\ell}$ of μ_ℓ to $\lim_{\substack{\longrightarrow \\ R' \in \mathcal{T}}} \frac{\widehat{R'}}{\tilde{H}'_{2\ell-1}}$. The valuation $\hat{\mu}_2$ will be the desired scalewise-birational extension $\hat{\nu}_-$ of $\mu_1 = \nu$.

We will have $\tilde{H}'_{2\ell} = \tilde{H}'_{2\ell+1}$.

Let $\widehat{\mathbf{R}} := \lim_{\substack{\longrightarrow \\ R' \in \mathcal{T}}} \widehat{R'}$ and $\mathbf{H}_i = \lim_{\substack{\longrightarrow \\ R' \in \mathcal{T}}} \tilde{H}'_i$. Then $\hat{\nu}_{2\ell+1}$ will be the trivial valuation of the field $\frac{\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell+1}}}{\mathbf{H}_{2\ell} \widehat{\mathbf{R}}_{\mathbf{H}_{2\ell+1}}}$.

We define $\tilde{H}'_{2r} = \tilde{H}'_{2r+1} = m' \widehat{R'}$ and let $\hat{\mu}_{2r+2} = \hat{\nu}_{2r+2}$ be the trivial valuation of the residue field k' .

Next, assume that $\ell \in \{1, \dots, r\}$, that the tree $\{\tilde{H}'_{2\ell+1}\} \supset \{H'_{2\ell+1}\}$ of prime ideals of $\widehat{R'}$ and a scalewise-birational extension $\hat{\mu}_{2\ell+2}$ of $\mu_{\ell+1}$ to $\lim_{\substack{\longrightarrow \\ R' \in \mathcal{T}}} \frac{\widehat{R'}}{\tilde{H}'_{2\ell+1}}$ are already constructed for $R' \in \mathcal{T}$ and that

$$\tilde{H}'_{2\ell+1} \cap R' = P'_\ell.$$

Define $\tilde{H}'_{2\ell} := \tilde{H}'_{2\ell+1}$ and let $\hat{\nu}_{2\ell+1}$ be the trivial valuation of the field $\frac{\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell+1}}}{\mathbf{H}_{2\ell} \widehat{\mathbf{R}}_{\mathbf{H}_{2\ell+1}}}$.

It remains to construct the ideals $\tilde{H}'_{2\ell-1} \subset \widehat{R'}$ and a scalewise-birational extension $\hat{\mu}_{2\ell}$ of μ_ℓ to $\frac{\widehat{R'}}{\tilde{H}'_{2\ell-1}}$ for R' in $\mathcal{T}$.

Consider the set $\mathcal{I}$ of all the trees of prime ideals $\{H'\}$ of $\{\widehat{R}'\}$ contained in $\tilde{H}'_{2\ell}$ and satisfying the condition

$$H' \cap R' = P'_{\ell-1}. \quad (3.1)$$

The collection $\mathcal{I}$ is not empty since the tree $\{H'_{2\ell-1}\}$ belongs to it by Proposition 2.1 (Proposition 3.1 of [8]). Let $\{\tilde{H}'_{2\ell-1}\}$ be a maximal (in the sense of inclusion) element of $\mathcal{I}$. Here by “maximal” we mean that there is no tree $\{I'\}$ in $\mathcal{I}$ containing $\{\tilde{H}'_{2\ell-1}\}$ such that $\{\tilde{H}'_{2\ell-1}\} \subsetneq I'$ for some I' in $\mathcal{I}$ and hence for all I' that are sufficiently far out in the tree. Such a maximal tree $\{\tilde{H}'_{2\ell-1}\}$ exists by Zorn’s Lemma.

By the induction assumption, the valuation $\hat{\mu}_{2\ell+2}$ is already defined on $\frac{\widehat{\mathbf{R}}}{\mathbf{H}_{2\ell+1}}$ and, for all $R' \in \mathcal{T}$, the inclusion $\text{gr}_{\mu_{\ell+1}} \frac{R'}{P'_\ell} \subset \text{gr}_{\hat{\mu}_{2\ell+2}} \frac{\widehat{R}'}{\tilde{H}'_{2\ell+1}}$ of graded algebras is scalewise-birational.

It remains to describe the valuation $\hat{\mu}_{2\ell}$ centered in $\frac{\widehat{R}'}{\tilde{H}'_{2\ell-1}}$, $R' \in \mathcal{T}$. We will first define $\hat{\nu}_{2\ell}$ and then define $\hat{\mu}_{2\ell}$ as $\hat{\nu}_{2\ell} \circ \hat{\mu}_{2\ell+2}$.

LEMMA 3.1. *Let μ be a rank one valuation, centered in a local noetherian domain (A, M) . Let*

$$\Psi = \mu(A \setminus \{0\}).$$

Then Ψ contains no infinite strictly increasing bounded sequences.

Proof. For an element $\beta \in \Psi$, let $\mathcal{P}_\beta := \mathcal{P}_\beta(A)$ denote the μ -ideal of A of value β . An infinite ascending sequence $\alpha_1 < \alpha_2 < \dots$ in Ψ , bounded above by an element $\beta \in \Psi$, would give rise to an infinite descending chain of ideals in $\frac{A}{\mathcal{P}_\beta}$. Thus it is sufficient to prove that $\frac{A}{\mathcal{P}_\beta}$ has finite length.

Let $\delta := \mu(M) (= \min(\Psi \setminus \{0\}))$. Since $\text{rk } \mu = 1$, there exists $n \in \mathbf{N}$ such that $\beta \leq n\delta$. Then $M^n \subset \mathcal{P}_\beta$, so there is a surjective homomorphism $\frac{A}{M^n} \twoheadrightarrow \frac{A}{\mathcal{P}_\beta}$. Thus $\frac{A}{\mathcal{P}_\beta}$ has finite length, as desired. $\square$

REMARK 2. See [4, Theorem 3.2] for a more general result about the absence of accumulation points in semigroups.

COROLLARY 3.2. *The semigroup $\nu \left(\frac{R'_{P'_\ell}}{P'_{\ell-1}} \setminus \{0\} \right)$ is isomorphic to $\mathbf{N}$ as an ordered set.*

Proof. Since ν_ℓ is a rank 1 valuation, centered in the local noetherian domain $\frac{R'_{P'_\ell}}{P'_{\ell-1}}$, the semigroup $\nu \left(\frac{R'_{P'_\ell}}{P'_{\ell-1}} \setminus \{0\} \right)$ contains no infinite strictly increasing bounded sequences. The corollary now follows from the fact that $\nu \left(\frac{R'_{P'_\ell}}{P'_{\ell-1}} \setminus \{0\} \right)$ is well ordered. $\square$

LEMMA 3.3. Fix a ring R' in $\mathcal{T}$ and a non-zero element $x \in \frac{\widehat{R'}_{\tilde{H}'_{2\ell}}}{\tilde{H}'_{2\ell-1} \widehat{R'}_{\tilde{H}'_{2\ell}}}$. Then

$$(x) \cap \frac{R'_{P'_\ell}}{P'_{\ell-1}} \neq (0).$$

Proof. Let $Q_1, \dots, Q_s$ be the set of minimal primes of (x) in $\frac{\widehat{R'}_{\tilde{H}'_{2\ell}}}{\tilde{H}'_{2\ell-1} \widehat{R'}_{\tilde{H}'_{2\ell}}}$. It is sufficient to prove that

$$Q_i \cap \frac{R'_{P'_\ell}}{P'_{\ell-1}} \neq (0) \quad \text{for all } i \in \{1, \dots, s\}. \quad (3.2)$$

Indeed, once (3.2) is established, let y_i be a non-zero element of $Q_i \cap \frac{R'_{P'_\ell}}{P'_{\ell-1}}$. Then there exists $N \in \mathbf{N}$ such that $0 \neq \left(\prod_{i=1}^s y_i \right)^N \in (x) \cap \frac{R'_{P'_\ell}}{P'_{\ell-1}}$, as desired.

To prove (3.2), we argue by contradiction. Assume that (x) has a minimal prime Q' in $\frac{\widehat{R'}_{\tilde{H}'_{2\ell}}}{\tilde{H}'_{2\ell-1} \widehat{R'}_{\tilde{H}'_{2\ell}}}$ satisfying

$$Q' \cap \frac{R'_{P'_\ell}}{P'_{\ell-1}} = (0).$$

For every ν -extension $R' \rightarrow R''$, there exists a minimal prime Q'' of (x) in $\frac{\widehat{R''}_{\tilde{H}''_{2\ell}}}{\tilde{H}''_{2\ell-1} \widehat{R''}_{\tilde{H}''_{2\ell}}}$ lying over Q' and satisfying

$$Q'' \cap \frac{R''_{P''_\ell}}{P''_{\ell-1}} = (0). \quad (3.3)$$

By Zorn's lemma, there exists a tree $\{P''\}$ of prime ideals of $\left\{ \frac{\widehat{R''}_{\tilde{H}''_{2\ell}}}{\tilde{H}''_{2\ell-1} \widehat{R''}_{\tilde{H}''_{2\ell}}} \right\}$ satisfying (3.3). Taking the preimage of $\{P''\}$ in $\{\widehat{R''}\}$ produces a tree $\{H''\}$ satisfying (3.1) and properly containing the tree $\{\tilde{H}''_{2\ell-1}\}$, contradicting the maximality of $\{\tilde{H}''_{2\ell-1}\}$. This completes the proof of the lemma. $\square$

For a positive element $\bar{\beta} \in \frac{\Delta_{\ell-1}}{\Delta_\ell}$ denote by $\widehat{\mathcal{P}}'_{\bar{\beta}, \ell}$ the ideal

$$\widehat{\mathcal{P}}'_{\bar{\beta}, \ell} = \frac{(\mathcal{P}'_{\bar{\beta}} + \tilde{H}'_{2\ell-1}) \widehat{R'}_{\tilde{H}'_{2\ell}}}{\tilde{H}'_{2\ell-1} \widehat{R'}_{\tilde{H}'_{2\ell}}}. \quad (3.4)$$

Since by the definition of $\tilde{H}'_{2\ell-1}$ we have $\bigcap_{\bar{\beta}} \widehat{\mathcal{P}}'_{\bar{\beta}, \ell} = (0)$, we can define the pseudo-valuation $\hat{\nu}_{2\ell}$ centered at $\frac{\widehat{R'}_{\tilde{H}'_{2\ell}}}{\tilde{H}'_{2\ell-1} \widehat{R'}_{\tilde{H}'_{2\ell}}}$ by

$$\hat{\nu}_{2\ell}(x) := \max \left\{ \bar{\beta} \in \left(\frac{\Delta_{\ell-1}}{\Delta_\ell} \right)_+ \mid x \in \widehat{\mathcal{P}}'_{\bar{\beta}, \ell} \right\}. \quad (3.5)$$

LEMMA 3.4. We have

$$\widehat{\mathcal{P}}'_{\bar{\beta}} \cap \frac{R'_{P'_\ell}}{P'_{\ell-1}} = \mathcal{P}'_{\bar{\beta}}. \quad (3.6)$$

Proof. The inclusion $\supset$ is clear from definitions. Let us prove the opposite inclusion.

By Zorn's lemma, there exist extensions $\tilde{\nu}$ of ν_ℓ to valuations centered in the ring $\frac{\widehat{R'}_{\widehat{H}'_{2\ell}}}{\widehat{H}'_{2\ell-1}\widehat{R'}_{\widehat{H}'_{2\ell}}}$.

Fix one such extension $\tilde{\nu}$ and let $\tilde{\mathcal{P}}'_\beta$ denote the greatest $\tilde{\nu}$ -ideal of $\frac{\widehat{R'}_{\widehat{H}'_{2\ell}}}{\widehat{H}'_{2\ell-1}\widehat{R'}_{\widehat{H}'_{2\ell}}}$ among those of value at least $\bar{\beta}$. We have

$$\widehat{\mathcal{P}}'_{\bar{\beta},\ell} \subset \tilde{\mathcal{P}}'_\beta, \quad (3.7)$$

so $\widehat{\mathcal{P}}'_{\bar{\beta},\ell} \cap \frac{R'_{P'_\ell}}{P'_{\ell-1}} \subset \tilde{\mathcal{P}}'_\beta \cap \frac{R'_{P'_\ell}}{P'_{\ell-1}} = \mathcal{P}'_\beta$, which proves the desired inclusion. $\square$

COROLLARY 3.5. *The restriction of the pseudo-valuation $\widehat{\nu}_{2\ell}$ to $\frac{R'_{P'_\ell}}{P'_{\ell-1}}$ coincides with ν_ℓ .*

REMARK 3. The inclusion (3.7) is equivalent to saying that for every $x \in \frac{\widehat{R'}_{\widehat{H}'_{2\ell}}}{\widehat{H}'_{2\ell-1}\widehat{R'}_{\widehat{H}'_{2\ell}}}$ we have $\tilde{\nu}(x) \leq \widehat{\nu}_{2\ell}(x)$. The next lemma shows that this inequality is, in fact, an equality.

Keep the notation of the above remark. Let $\bar{\beta} = \widehat{\nu}_{2\ell}(x)$.

LEMMA 3.6. *We have*

$$\tilde{\nu}(x) = \bar{\beta}. \quad (3.8)$$

Proof. Take a non-zero element $y \in (x) \cap \frac{R'_{P'_\ell}}{P'_{\ell-1}}$ (such an element exists by Lemma 3.3). Let $u \in \frac{\widehat{R'}_{\widehat{H}'_{2\ell}}}{\widehat{H}'_{2\ell-1}\widehat{R'}_{\widehat{H}'_{2\ell}}}$ be such that $y = ux$. Since $\widehat{\nu}_{2\ell}$ is a pseudo-valuation extending ν_ℓ , we have

$$\nu_\ell(y) = \widehat{\nu}_{2\ell}(y) = \widehat{\nu}_{2\ell}(xu) \geq \bar{\beta} + \widehat{\nu}_{2\ell}(u) \geq \tilde{\nu}(x) + \tilde{\nu}(u) = \tilde{\nu}(y) = \nu_\ell(y), \quad (3.9)$$

where the first inequality is given by the ultrametric triangle rule and the second follows from Remark 3, applied separately to x and u . Thus all the inequalities in (3.9) are equalities and the result follows. $\square$

COROLLARY 3.7.

- (i) *The pseudo-valuation $\widehat{\nu}_{2\ell}$ is, in fact, a valuation.*
- (ii) *The valuation $\widehat{\nu}_{2\ell}$ is the unique valuation extending ν_ℓ to $\frac{\widehat{R'}_{\widehat{H}'_{2\ell}}}{\widehat{H}'_{2\ell-1}\widehat{R'}_{\widehat{H}'_{2\ell}}}$.*

REMARK 4. Since the construction above is valid for all R'' in $\mathcal{T}$, $\widehat{\nu}_{2\ell}$ extends naturally to a valuation $\widehat{\nu}_{2\ell} : \frac{\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell+1}}}{\widehat{\mathbf{H}}_{2\ell+1}} \setminus \{0\} \rightarrow \frac{\Delta_{\ell-1}}{\Delta_\ell}$ (which we still denote by $\widehat{\nu}_{2\ell}$). The valuation $\widehat{\nu}_{2\ell}$ is centered in the local ring $\frac{\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell+1}}}{\widehat{\mathbf{H}}_{2\ell+1}}$. Moreover, $\widehat{\nu}_{2\ell}$ is the unique valuation extending ν_ℓ with this property.

Let

$$\mathbf{P}_{\bar{\beta}} := \lim_{R' \in \mathcal{T}} \frac{\mathcal{P}'_{\bar{\beta}} R'_{P'_\ell}}{P'_{\ell-1} R'_{P'_\ell}},$$

$$\widehat{\mathbf{P}}_{\bar{\beta},\ell} := \lim_{\substack{\longrightarrow \\ R' \in \mathcal{T}}} \widehat{\mathcal{P}}'_{\bar{\beta},\ell},$$

and similarly for $\mathbf{P}_{\bar{\beta}}^+$ and $\widehat{\mathbf{P}}_{\bar{\beta},\ell}^+$. Recall that

$$\mathbf{m}_{\ell} = \lim_{\substack{\longrightarrow \\ R' \in \mathcal{T}}} P'_{\ell}.$$

By definitions (see (3.4)), for all $\bar{\beta} \in \left(\frac{\Delta_{\ell}-1}{\Delta_{\ell}}\right)_+$, we have a natural surjective homomorphism

$$\lambda_{\bar{\beta}} : \frac{\mathcal{P}'_{\bar{\beta}}}{\mathcal{P}'_{\bar{\beta}}^+} \otimes_{\kappa(P'_{\ell})} \kappa(\tilde{H}'_{2\ell}) \longrightarrow \frac{\widehat{\mathcal{P}}'_{\bar{\beta},\ell}}{\widehat{\mathcal{P}}'_{\bar{\beta},\ell}^+} \quad (3.10)$$

of $\kappa(\tilde{H}'_{2\ell})$ -vector spaces. Passing to direct limits as R' runs over the tree $\mathcal{T}$, we obtain a natural surjective homomorphism

$$\lambda_{\bar{\beta}} : \frac{\mathbf{P}_{\bar{\beta}}}{\mathbf{P}_{\bar{\beta}}^+} \otimes_{\kappa(\mathbf{P}_{\ell})} \kappa(\mathbf{H}_{2\ell}) \longrightarrow \frac{\widehat{\mathbf{P}}_{\bar{\beta},\ell}}{\widehat{\mathbf{P}}_{\bar{\beta},\ell}^+} \quad (3.11)$$

of $\kappa(\mathbf{H}_{2\ell})$ -vector spaces.

REMARK 5.

(i) We have

$$\frac{(R_{\nu})_{\mathbf{m}_{\ell}}}{\mathbf{m}_{\ell-1}(R_{\nu})_{\mathbf{m}_{\ell}}} = R_{\nu_{\ell}}.$$

The ideal $\mathbf{P}_{\bar{\beta}} \subset \frac{(R_{\nu})_{\mathbf{m}_{\ell}}}{\mathbf{m}_{\ell-1}(R_{\nu})_{\mathbf{m}_{\ell}}}$ is the ν_{ℓ} -ideal of value $\bar{\beta}$; in particular, $\mathbf{P}_{\bar{\beta}}$ is principal, generated by any element $x \in \frac{(R_{\nu})_{\mathbf{m}_{\ell}}}{\mathbf{m}_{\ell-1}(R_{\nu})_{\mathbf{m}_{\ell}}}$ such that $\nu_{\ell}(x) = \bar{\beta}$.

(ii) By definitions (see (3.4)), we have

$$\widehat{\mathbf{P}}_{\bar{\beta},\ell} = \mathbf{P}_{\bar{\beta}} \frac{\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}{\mathbf{H}_{2\ell-1} \widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}. \quad (3.12)$$

Hence the ideal $\widehat{\mathbf{P}}_{\bar{\beta},\ell}$ is also principal.

By Remark 5, both $\kappa(\mathbf{H}_{2\ell})$ -vector spaces in (3.11) are one-dimensional.

COROLLARY 3.8. *The map (3.11) (and hence also (3.10)) is an isomorphism.*

COROLLARY 3.9. *The graded algebra*

$$gr_{\nu_{\ell}} \frac{(R_{\nu})_{\mathbf{m}_{\ell}}}{\mathbf{m}_{\ell-1}(R_{\nu})_{\mathbf{m}_{\ell}}} \otimes_{\kappa(\mathbf{m}_{\ell})} \kappa(\mathbf{H}_{2\ell}) \cong \bigoplus_{\bar{\beta} \in \left(\frac{\Delta_{\ell}-1}{\Delta_{\ell}}\right)_+} \frac{\widehat{\mathbf{P}}_{\bar{\beta},\ell}}{\widehat{\mathbf{P}}_{\bar{\beta},\ell}^+} = gr_{\widehat{\nu}_{2\ell}} \frac{\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}{\mathbf{H}_{2\ell-1} \widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}$$

is an integral domain.

The extension $\widehat{\mu}_{2\ell}$ of μ_{ℓ} to $\frac{\widehat{\mathbf{R}}}{\mathbf{H}_{2\ell-1}}$ is defined by $\widehat{\mu}_{2\ell} = \widehat{\nu}_{2\ell} \circ \widehat{\mu}_{2\ell+2}$. This completes the definition of $\widehat{\mu}_{2\ell}$, $\ell \in \{1, \dots, r\}$, by descending recursion on ℓ .

LEMMA 3.10. *The natural inclusion*

$$\mathrm{gr}_{\mu_\ell} \frac{R}{P_{\ell-1}} \hookrightarrow \mathrm{gr}_{\widehat{\mu}_{2\ell}} \frac{\widehat{R}}{\widehat{H}_{2\ell-1}} \quad (3.13)$$

of graded algebras is scalewise-birational.

Proof. In the proof that follows we will keep in mind the following commutative diagram

$$\begin{array}{ccccc} & \mathrm{gr}_{\mu_\ell} \frac{R'}{P'_{\ell-1}} & \hookrightarrow & \mathrm{gr}_{\widehat{\mu}_{2\ell}} \frac{\widehat{R}'}{\widehat{H}'_{2\ell-1}} & (3.14) \\ & \uparrow & & \uparrow & \\ \mathrm{gr}_{\mu_{\ell+1}} \frac{R}{P_\ell} & \hookrightarrow & \mathrm{gr}_{\mu_{\ell+1}} \frac{R'}{P'_\ell} & \hookrightarrow & \mathrm{gr}_{\widehat{\mu}_{2\ell+2}} \frac{\widehat{R}'}{\widehat{H}'_{2\ell}} \end{array}$$

of natural inclusions, valid for all R' in $\mathcal{T}$. Here the lower row is a composition of two birational injections of graded algebras, where the second inclusion is scalewise-birational by the induction hypothesis, and the top row is an inclusion of graded algebras whose scalewise birationality we want to prove.

Take a homogeneous element $\bar{x} \in \mathrm{gr}_{\widehat{\mu}_{2\ell}} \frac{\widehat{R}}{\widehat{H}_{2\ell-1}}$ and let x be a representative of $\bar{x}$ in $\frac{\widehat{R}}{\widehat{H}_{2\ell-1}}$. Let $\bar{\beta} = \widehat{\mu}_{2\ell}(x)$. Take an element $y \in \mathbf{P}_{\bar{\beta}}$ and $u \in \frac{\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}{\mathbf{H}_{2\ell-1} \widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}} \setminus \mathbf{H}_{2\ell} \frac{\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}{\mathbf{H}_{2\ell-1} \widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}$ such that $x = yu^{-1}$ (the existence of y and u follows from Remark 5 (ii)).

Let R' be such that $y \in \frac{R'_{P_\ell}}{P'_{\ell-1} R'_{P_\ell}}$ and $u \in \frac{\widehat{R}'_{\widehat{H}'_{2\ell}}}{\widehat{H}'_{2\ell-1} R'_{\widehat{H}'_{2\ell}}} \setminus \widehat{H}'_{2\ell} \frac{\widehat{R}'_{\widehat{H}'_{2\ell}}}{\widehat{H}'_{2\ell-1} R'_{\widehat{H}'_{2\ell}}}$. Write $y = \frac{y_1}{y_2}$ with $y_1, y_2 \in \frac{R}{P_{\ell-1}}$. Write u as $u = \frac{w}{s}$, where

$$w, s \in \frac{\widehat{R}'}{\widehat{H}'_{2\ell-1}} \setminus \frac{\widehat{H}'_{2\ell}}{\widehat{H}'_{2\ell-1}} \quad (3.15)$$

for some R' in $\mathcal{T}$.

Let $\bar{\cdot}$ denote the operation of taking the natural image of an element of $\frac{\widehat{\mathbf{R}}}{\mathbf{H}_{2\ell-1}}$ in $\mathrm{gr}_{\widehat{\mu}_{2\ell}} \frac{\widehat{\mathbf{R}}}{\mathbf{H}_{2\ell-1}}$, its initial form with respect to the filtration defined by $\widehat{\mu}_{2\ell}$. The inclusion (3.15) implies that $\bar{w}, \bar{s} \in \mathrm{gr}_{\widehat{\mu}_{2\ell+2}} \frac{\widehat{R}'}{\widehat{H}'_{2\ell}}$. In view of the birationality of the second row of diagram (3.14), there exist $\bar{w}_1, \bar{w}_2, \bar{s}_1, \bar{s}_2 \in \mathrm{gr}_{\mu_{\ell+1}} \frac{R}{P_\ell}$ satisfying $\bar{w} = \frac{\bar{w}_1}{\bar{w}_2}$ and $\bar{s} = \frac{\bar{s}_1}{\bar{s}_2}$.

Putting together all of the above, we obtain

$$\bar{x}(\bar{y}_2 \bar{w}_1 \bar{s}_2) \in \mathrm{gr}_{\mu_\ell} \frac{R}{P_{\ell-1}}$$

and $\bar{y}_2 \bar{w}_1 \bar{s}_2 \in \mathrm{gr}_{\mu_\ell} \frac{R}{P_{\ell-1}}$, the latter containment implying that

$$\mathrm{ord}(\bar{y}_2 \bar{w}_1 \bar{s}_2) \in \Delta_{\ell-1}.$$

This completes the proof of the lemma. $\square$

We put $\hat{\nu}_- = \widehat{\mu}_2 = \hat{\nu}_2 \circ \cdots \circ \hat{\nu}_{2r}$, where for each ℓ the valuation $\hat{\nu}_{2\ell}$ is centered in the local ring $\frac{\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}{\mathbf{H}_{2\ell-1} \widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}$.

This completes the construction of the valuation $\hat{\nu}_-$.

The scalewise birationality of the extension $\hat{\nu}_-$ of ν to the ring $\frac{\widehat{R}}{\widehat{H}_1}$ is given by Lemma 3.10, applied with $\ell = 1$. This completes the proof of Theorem 1.1. $\square$

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