ON ALGEBRAICALLY MAXIMAL VALUED FIELDS THAT ARE NOT DEFECTLESS

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Abstract. An example originally given by F. Delon shows the existence of an algebraically maximal discretely valued field of characteristic $p > 0$ which admits purely inseparable extensions of degree p^2 with defect p. These extensions are not generated by a single element. Using a trick introduced in an earlier paper of the author, we construct algebraically maximal valued fields, of characteristic p as well as of characteristic 0, which admit separable extensions of degree p^2 with defect p. They are of rank 2 and it is an open question whether such examples having rank 1 exist.

1. Introduction

The notions and notations we will use will be introduced in Section 2.

Françoise Delon gave an example that shows that algebraically maximal valued fields are not necessarily defectless (see [2], Exemple 1.51). A corrected and expanded version was presented in [7, Example 3.25]. We reproduce it in Section 3 and fill a gap that appeared in its exposition in [7].

For what follows, take a prime p. Example 3.1 proves:

Theorem 1.1. *There exists a discretely valued algebraically maximal field* (L, v) *of characteristic* $p > 0$ *which is not inseparably defectless and admits a purely inseparable extension of degree* p *which is not an algebraically maximal field. In particular, the property "algebraically maximal" does not imply "defectless".*

The question arises whether there are also examples of algebraically maximal fields which admit separable (and hence simple) defect extensions. Using a trick already employed in [7, Example 3.18], we will construct such examples in Section 4, based on which we prove:

Theorem 1.2. *There exists an algebraically maximal field* (L, v) *with residue characteristic* $p > 0$ *which admits a separable extension of degree* p^2 *with defect* p*, and with intermediate fields of degree* p *over* L *which are not algebraically maximal fields. The field* L *can be chosen of characteristic* 0 *as well as of characteristic* p*. In particular, the property "algebraically maximal" does not imply "separably defectless" and is not preserved under finite separable extensions.*

The valuations in the examples we give to prove this theorem have rank 2.

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Open Problem: Are there algebraically maximal fields of rank 1 which admit separable defect extensions?

2. Preliminaries

For a valued field (K,v), we denote its value group by vK, its residue field by Kv, and its valuation ring by $\mathcal{O}_K$. By $(L|K,v)$ we denote an extension $L|K$ with valuation v on L, where K is endowed with the restriction of v. In this case, there are induced embeddings of vK in vL and of Kv in Lv. The extension $(L|K,v)$ is called **immediate** if these embeddings are onto. A valued field (K,v) is called **algebraically maximal** if it does not admit nontrivial immediate algebraic extensions, and it is called **maximal** if it does not admit any nontrivial immediate extensions.

If K is a field of characteristic $p>0$ and the extension $K|K^p$ is finite, then there is $k\geq 0$ such that $[K:K^p]=p^k$; we then take the **p-degree of** K (also called **degree of inseparability**) to be k.

A valued field (K,v) is called **henselian** if each algebraic extension $L|K$ is **unibranched**, that is, the extension of v to L is unique.

If $(L|K,v)$ is a finite unibranched extension, then by the Lemma of Ostrowski ([9, Corollary to Theorem 25, Section G, p. 78]),

$$[L:K] \;=\; \tilde{p}^{\nu}\cdot(vL:vK)[Lv:Kv]\,, \tag{1}$$

where ν is a nonnegative integer and $\tilde{p}$ the **characteristic exponent** of Kv, that is, $\tilde{p}=\operatorname{char}Kv$ if it is positive and $\tilde{p}=1$ otherwise. The factor $d(L|K,v):=\tilde{p}^{\nu}$ is the **defect** of the extension $(L|K,v)$. If $d(L|K,v)=1$, then the extension $(L|K,v)$ is called **defectless**; otherwise we call it a **defect extension**. A valued field (K,v) is a **separably defectless field** if every finite unibranched separable extension of (K,v) is defectless, and a **defectless field** if every finite unibranched extension of (K,v) is defectless; note that this is always the case if $\operatorname{char}Kv=0$. An arbitrary valued field is called an **inseparably defectless field** if every finite purely inseparable extension is defectless.

The defect is multiplicative: if $(L|K,v)$ and $(M|L,v)$ are finite unibranched extensions, then

$$\mathrm{d}(M|K,v)=\mathrm{d}(M|L,v)\cdot\mathrm{d}(L|K,v) \tag{2}$$

(see [7, Equation (4)]).

For a valued field (K,v) and a finite field extension $L|K$, the **Fundamental Inequality** (see (17.5) of [3] or Theorem 19 on p. 55 of [9]) states that there are finitely many extensions of v from K to L, and

$$[L:K] \;\geq\; \sum_{i=1}^{\mathrm{g}}(v_iL:vK)[Lv_i:Kv]\,, \tag{3}$$

where $v_1,\ldots,v_{\mathrm{g}}$ are the distinct extensions.

For a field K of characteristic $p>0$, we will denote its perfect hull by K^{1/p^∞}. An arbitrary field extension L of K is called **separable** if it is linearly disjoint over K from K^{1/p^∞}. If $L|K$ is algebraic, then this definition coincides with the usual definitions of separability.

Lemma 2.1. *An arbitrary field extension* L *of a field* K *of characteristic* $p > 0$ *is separable if and only if it is linearly disjoint over* K *from* $K^{1/p} = K(a^{1/p} \mid a \in K)$.

Proof. If the extension L of K is separable, then by definition it is linearly disjoint over K from K^{1/p^∞} and thus also from $K^{1/p}$. For the converse, assume that L is not linearly disjoint over K from K^{1/p^∞}. Then there are K-linearly independent elements $x_1, \ldots, x_n \in L$ which are not K^{1/p^∞}-linearly independent. Choose m minimal such that there are $y_1, \ldots, y_n \in K^{1/p^m} = K(a^{1/p^m} \mid a \in K)$ with $\sum_i x_i y_i = 0$. Then $m \geq 1$, and $x_1^{p^{m-1}}, \ldots, x_n^{p^{m-1}}$ are K-linearly independent. (Otherwise, we would have a non-trivial relation $\sum_i x_i^{p^{m-1}} z_i = 0$ with $z_i \in K$, hence $\sum_i x_i z_i^{1/p^{m-1}} = 0$, contradicting the minimality of m.) But $x_1^{p^{m-1}}, \ldots, x_n^{p^{m-1}}$ are not $K^{1/p}$-linearly independent since $\sum_i x_i^{p^{m-1}} y_i^{p^{m-1}} = 0$ with $y_i^{p^{m-1}} \in K^{1/p}$. This proves that $L|K$ is not linearly disjoint over K from $K^{1/p}|K$. □

3. A BASIC EXAMPLE

Example 3.1. We consider $\mathbb{F}_p((t))$ with its t-adic valuation v_t. Since $\mathbb{F}_p((t))$ has uncountable cardinality, while that of $\mathbb{F}_p(t)$ is countable, the extension $\mathbb{F}_p((t))|\mathbb{F}_p(t)$ has infinite transcendence degree, we can choose elements $x, y \in \mathbb{F}_p((t))$ which are algebraically independent over $\mathbb{F}_p(t)$. We set

$$s := x^p + ty^p \quad \text{and} \quad K := \mathbb{F}_p(t,s)\,.$$

We note that $K^{1/p} = \mathbb{F}_p(t^{1/p}, s^{1/p}) = K(t^{1/p}, s^{1/p})$. The elements t, s are algebraically independent over $\mathbb{F}_p$. Consequently, the p-degree of K is 2. We define L_0 to be the relative algebraic closure of K in $\mathbb{F}_p((t))$. Then $L_0(t^{1/p}, s^{1/p}) \subseteq L_0^{1/p}$.

We are going to show that L_0 is linearly disjoint over K from $K^{1/p}$ and the p-degree of L_0 is again 2. Since the elements $1, t^{1/p}, \ldots, t^{(p-1)/p}$ are linearly independent over $\mathbb{F}_p((t))$, the same holds over L_0. Hence, the elements $1, t, \ldots, t^{p-1}$ are linearly independent over L_0^p. Now if $s^{1/p}$ were an element of $L_0(t^{1/p})$, then it could be written in a unique way as an L_0-linear combination of $1, t^{1/p}, \ldots, t^{(p-1)/p}$, and s could be written in a unique way as an L_0^p-linear combination of $1, t, \ldots, t^{p-1}$. But this is not possible since $s = x^p + ty^p$ and x, y are transcendental over L_0. Hence, L_0 is linearly disjoint over K from $K^{1/p}$ and the p-degree of L_0 is again 2; more precisely,

$$L_0^{1/p} = L_0(t^{1/p}, s^{1/p}) \quad \text{with} \quad [L_0^{1/p} : L_0] = p^2\,.$$

From Lemma 2.1 it follows that the extension $L_0|K$ is separable.

Since $s^{1/p} = x + t^{1/p} y \in \mathbb{F}_p(t^{1/p}, x, y) \subset \mathbb{F}_p((t^{1/p}))$, we have $L_0^{1/p} \subset \mathbb{F}_p((t^{1/p}))$. Extending v_t to $\mathbb{F}_p((t^{1/p}))$, we obtain that

$$L_0^{1/p} v_t \subseteq \mathbb{F}_p((t^{1/p})) v_t = \mathbb{F}_p = K v_t \subseteq L_0 v_t \subseteq L_0^{1/p} v_t\,.$$

Further,

$$v_t L_0^{1/p} \subseteq v_t \mathbb{F}_p((t^{1/p})) = \frac{1}{p} v_t \mathbb{F}_p((t)) = v_t \mathbb{F}_p(t^{1/p}) \subseteq v_t L_0^{1/p}\,,$$

hence equality holds everywhere. Moreover, $v_t L_0 \subseteq v_t \mathbb{F}_p((t)) = v_t \mathbb{F}_p(t) \subseteq v_t L_0$, showing that $v_t L_0 = v_t \mathbb{F}_p((t))$. Consequently,

$$(v_t L_0^{1/p} : v_t L_0) = p \quad \text{and} \quad [L_0^{1/p} v_t : L_0 v_t] = 1$$

As a relatively algebraically closed subfield of the henselian field $(\mathbb{F}_p((t)),v_t)$, also (L_0,v_t) is henselian. Thus the extension $(L_0^{1/p}|L_0,v_t)$ is unibranched and consequently has defect p.

On the other hand, $\mathbb{F}_p((t))$ is the completion of (L_0,v_t) since it is already the completion of $\mathbb{F}_p(t)\subseteq L_0$. This shows that $\mathbb{F}_p((t))$ is the unique maximal immediate extension of L_0 (up to valuation preserving isomorphism over L_0). If L_0 would admit a proper immediate algebraic extension L_1, then a maximal immediate extension of L_1 would also be a maximal immediate extension of L_0 and would thus be isomorphic over L_0 to $\mathbb{F}_p((t))$. But we have chosen L_0 to be relatively algebraically closed in $\mathbb{F}_p((t))$. This proves that (L_0,v) must be algebraically maximal. Hence

$$(L_0(s^{1/p})|L_0,v_t) \quad \text{and} \quad (L_0(t^{1/p})|L_0,v_t)\,,$$

cannot be immediate and, being of prime degree, are therefore defectless. Thus the defect of $L_0^{1/p}|L_0$ implies by multiplicativity (2) that both

$$(L_0^{1/p}|L_0(s^{1/p}),v_t) \quad \text{and} \quad (L_0^{1/p}|L_0(t^{1/p}),v_t)$$

must have defect p. Consequently, $(L_0(s^{1/p},v_t)$ and $(L_0(t^{1/p},v_t)$ are not algebraically maximal. ◇

We summarize the properties of this example, thereby adjusting the notation for later use.

Proposition 3.2. *There exists a discretely valued algebraically maximal field (L_0,v_0) of characteristic $p>0$ and purely inseparable defectless extensions $(L_0(a_0)|L_0,v_0)$ and $(L_0(b_0)|L_0,v_0)$ of degree p such that the unibranched extension $(L_0(a_0,b_0)|L_0,v_0)$ of degree p^2 has defect p, as $(v_0L_0(a_0,b_0):v_0L_0)=p$ and $[L_0(a_0,b_0)v_0:L_0v_0]=1$, and neither $(L_0(a_0),v_0)$ nor $(L_0(b_0),v_0)$ is an algebraically maximal field.* □

Remark 3.3. In [7, Example 3.25] it is stated that the relative algebraic closure L_0 of (K,v_t) in $\mathbb{F}_p((t))$ is a separable extension of K and therefore is the henselization of K. However, the proof on page 295 of [7] contains a gap (but this does not affect the results on the previous pages). In fact, it turns out that the use of the auxiliary field $\mathbb{F}_p(t,x,y)$ which contains K is not even necessary. Indeed, we have shown that the extension $L_0|K$ is separable. Since by definition, L_0 is relatively closed in the henselian field $\mathbb{F}_p((t))$, it is itself henselian and thus contains the henselization K^h of K. Now $\mathbb{F}_p((t))$ is the completion of K^h since it is already the completion of $\mathbb{F}_p(t)\subseteq K^h$. Since a henselian field is relatively separable-algebraically closed in its completion (cf. [8], Theorem 32.19) and $L_0|K^h$ is separable, it follows that $L_0=K^h$.

Nevertheless, the field $F:=\mathbb{F}_p(t,x,y)$ is useful for showing that the extension $\mathbb{F}_p((t))|L_0$ is not separable. We have $s^{1/p}=x+t^{1/p}y\in F(t^{1/p})$. Hence, $F.K^{1/p}=F(t^{1/p},s^{1/p})=F(t^{1/p})$ and $[F.K^{1/p}:F]=[F(t^{1/p}):F]\leq p<p^2=[K^{1/p}:K]$, that is, F is not linearly disjoint over K from $K^{1/p}$ and thus not separable, according to Lemma 2.1. Since $F\subset\mathbb{F}_p((t))$, it follows that $\mathbb{F}_p((t))|K$ is not separable. Since $L_0|K$ is separable, this implies that $\mathbb{F}_p((t))|L_0$ is not separable.

4. Examples with composite valuations

Lemma 4.1. *Take any field L_0 of positive characteristic. There exist henselian defectless discretely valued fields (L,w) with residue field L_0. They can be chosen such that either* $\operatorname{char} L = 0$, *or* $\operatorname{char} L = \operatorname{char} L_0$.

Proof. For $\operatorname{char} L = 0$: Take an extension of $(\mathbb{Q}, v_p)$, where v_p denotes the p-adic valuation, with value group equal to $v_p\mathbb{Q}$ and residue field L_0. For the construction of such extensions, see [5, Theorem 2.14]. Let (L, v_p) be the henselization of this field. Since (L, v_p) is henselian discretely valued of characteristic 0, it is a defectless field by [4, Theorem 8.32]. Alternatively, one can also take the completion in place of the henselization; as the valuation is still discrete, this field is maximal and therefore a henselian defectless field (see the discussion at the beginning of Section 4 in [1]).

For $\operatorname{char} L = \operatorname{char} L_0$: Take an element z transcendental over L_0, the z-adic valuation v_z on $L_0(z)$, and (L, v_z) to be the henselization of $(L_0(z), v_z)$. Then (L, v_z) is henselian discretely valued with residue field L_0, and by [6, Theorem 1.1], it is a defectless field. $\square$

If w is a valuation on a field L and v_0 a valuation on the residue field Lw, then one can define the **composition** $w \circ v_0$ which is a valuation on L with residue field $(Lw)v_0$ as follows. If $\mathcal{O}_w$ is the valuation ring of w on L and $\mathcal{O}_{v_0}$ is the valuation ring of v_0 on Lw, then the preimage of $\mathcal{O}_{v_0}$ under the residue map $\mathcal{O}_w \ni a \mapsto aw$ is a valuation ring contained in $\mathcal{O}_w$ and we let $v := w \circ v_0$ be the valuation associated with it (it is unique up to equivalence of valuations). Then, modulo canonical isomorphisms, $v_0(Lw)$ can be viewed as a convex subgroup of vL and wL can be viewed as the quotient $vL/v_0(Lw)$.

Now let $(L'|L, v)$ be a finite extension. Then w extends to L' and v_0 to $L'w$ in such a way that $v_0(L'w)$ is the convex hull of $v_0(Lw)$ and wL' can be identified with $vL'/v_0(L'w)$. In this situation, we have:

Lemma 4.2. *If $wL' = wL$, then $(vL' : vL) = (v_0(L'w) : v_0(Lw))$.*

Proof. The equality $wL' = wL$ means that the natural embedding

$$vL/v_0(Lw) \;=\; wL \;\hookrightarrow\; wL' \;=\; vL'/v_0(L'w)$$

is onto. It follows that $vL' = vL + v_0(L'w)$, which in turn implies our statement. $\square$

Lemma 4.3. *Take (L_0, v_0) as in Lemma 3.2, and (L, w) as in Lemma 4.1. Set $v := w \circ v_0$. Then (L, v) is algebraically maximal.*

Proof. Since (L, w) is henselian defectless, it is algebraically maximal. The composition $w \circ v_0$ of two algebraically maximal valuations w and v_0 is again algebraically maximal (this is well known and the proof is straightforward). $\square$

Lemma 4.4. *Let (L, v) be as in the previous lemma. Then there are elements a, b in the separable-algebraic closure of L such that $[L(a) : L] = [L(b) : L] = p$, $L(a)w = L_0(a_0)$, and $L(b)w = L_0(b_0)$.*

Proof. Take $c, d \in L$ such that $cw = a_0^p \in L_0$ and $dw = b_0^p \in L_0$. If $\operatorname{char} L = 0$, then take a to be a p-th root of c and b to be a p-th root of d. If $\operatorname{char} L = \operatorname{char} L_0 = p$, then take a to be a root of the polynomial $X^p - rX - c$ and b to be a root of the polynomial $X^p - rX - d$ for some $r \in L \setminus \{0\}$ with $wr > 0$. Then in both cases, a and b are separable over L with $aw = a_0$ and $bw = b_0$. It follows that

$$p \;\geq\; [L(a) : L] \;\geq\; [L(a)w : Lw] \;\geq\; [L_0(a_0) : L_0] \;=\; p\,.$$

Hence equality holds everywhere, which proves that $[L(a) : L] = p$ and $L(a)w = L_0(a_0)$. The proof for b in place of a is similar. □

Now we are ready for the

Proof of Theorem 1.2: We shall prove that the valued field (L, v) of the previous lemma has the properties stated in Theorem 1.2. As the extensions $L(a)|L$ and $L(b)|L$ are separable, so is the extension $L(a, b)|L$. Since $a_0, b_0 \in L(a, b)w$, we have

$$p^2 \;\geq\; [L(a,b) : L] \;\geq\; [L(a,b)w : Lw] \;\geq\; [L_0(a_0,b_0) : L_0] \;=\; p^2\,,$$

hence equality holds everywhere, showing that $[L(a, b) : L] = p^2$ and $L(a, b)w = L_0(a_0, b_0)$, so that $L(a, b)v = L_0(a_0, b_0)v_0$. On the other hand, $wL(a, b) = wL$ by the Fundamental Equality (3) since $[L(a, b) : L] = [L(a, b)w : Lw]$. Further, by Proposition 3.2, $(v_0L_0(a_0, b_0) : v_0L_0) = p$ and $[L_0(a_0, b_0)v_0 : L_0v_0] = 1$. Hence

$$(vL(a,b) : vL) \;=\; (v_0(L(a,b)w) : v_0(Lw)) \;=\; (v_0L_0(a_0,b_0) : v_0L_0) = p\,,$$

where we have used Lemma 4.2, and

$$[L(a,b)v : Lv] = [L_0(a_0,b_0)v_0 : L_0v_0] \;=\; 1\,.$$

So the extension $(L(a, b)|L, v)$ has defect p.

Finally, $[L(a) : L] = p$ and since $wL(a, b) = wL$, we also have $wL(a) = wL$. Hence by Lemma 4.2, $(vL(a) : vL) = (v_0(L(a)w) : v_0(Lw)) = (v_0L_0(a_0) : v_0L_0) = p$, showing that the extension $(L(a)|L, v)$ is defectless. Since the defect is multiplicative, it follows that $(L(a, b)|L(a), v)$ has defect p, which shows that $(L(a), v)$ is not algebraically maximal. The same proof works for b in place of a. □

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